Geometric Interpretation of Electromagnetism in a Gravitational Theory with Torsion and Spinorial Matter

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Chapter 1

Introduction

Einstein's general relativistic theory of gravitation (see e.g. [Mis 73] and references therein) is based on a semi-Riemannian geometry. This spacetime geometry is characterized by a pseudo-Riemannian metric $g_{\mu\nu}$ and a linear connection $\Gamma^{\alpha}{}_{\mu\beta}$, which is compatible with the metric,

$$\nabla_{\mu} g_{\alpha\beta} = \partial_{\mu} g_{\alpha\beta} - \Gamma^{\gamma}{}_{\mu\alpha} g_{\gamma\beta} - \Gamma^{\gamma}{}_{\mu\beta} g_{\alpha\gamma} = 0 \; , \tag{1.1}$$

and has vanishing torsion

$$T^{\alpha}{}_{\mu\beta} := \Gamma^{\alpha}{}_{\mu\beta} - \Gamma^{\alpha}{}_{\beta\mu} = 0 \; . \tag{1.2}$$

These two requirements uniquely determine a special connection, the Levi–Civita connection

$$\Gamma^{\alpha}{}_{\mu\beta} = \{{}^{\alpha}{}_{\mu\beta}\} := \frac{1}{2} g^{\alpha\epsilon} \left(\partial_{\mu} g_{\epsilon\beta} + \partial_{\beta} g_{\epsilon\mu} - \partial_{\epsilon} g_{\mu\beta} \right) \; . \tag{1.3}$$

Within this semi-Riemannian geometry, the mass-energy of matter influences the spacetime via the Einstein's field equation

$$\frac{1}{k} G^{*}_{\alpha\beta} = T^{m}_{\alpha\beta} \; , \tag{1.4}$$

where $G^{*}_{\alpha\beta}$ is the Einstein–tensor, $k = 8\pi G/c^4$, and $T^{m}_{\alpha\beta}$ is the energy-momentum tensor of matter. Since $G^{*}_{\alpha\beta}$ is a tensor built from the Riemann curvature, the Einstein equation describes how the mass-energy of matter curves the spacetime. As far as macroscopic bulk matter is considered, the physical property of the matter is sufficiently characterized by this energy-momentum equation. However, on the microscopic level, the elementary particles are described by quantum mechanics and are not only characterized by mass, but also by spin. Therefore, to consider gravitational phenomena also on the microscopic level and to make general relativity more compatible with quantum mechanics, it seems necessary to take into account the influence of spin on the geometry of spacetime.

This aim is achieved in the so-called Einstein–Cartan theory by the use of an extended geometry. The crucial feature of this geometry is the non-vanishing torsion of the linear connection. Torsion was originally introduced by E. Cartan [Car 22, Car 23-25], who also developed a general relativistic theory with torsion, which contained the rudiments of the Einstein–Cartan theory. Surprisingly, although the spin of elementary particles was unknown at that time, he expected a connection between torsion and the intrinsic angular-momentum properties of matter.[1] The Einstein–Cartan theory in its final form was developed by many authors [Kib 61, Sci 62, Heh 66, Tra 71,72, Heh 76]. For a review see [Heh 76]. The geometry of the spacetime is now described by the so-called Riemann–Cartan geometry, in which the connection ${\Gamma^\alpha}_{\mu\beta}$ is only required to be compatible with the metric (1.1), but is allowed to have non-vanishing torsion contrary to the torsionless Levi–Civita connection. Due to the metricity condition (1.1) the connection now becomes [Heh 76]

$$ {\Gamma^\alpha}_{\mu\beta} = \{{}^\alpha{}_{\mu\beta}\} + \frac{1}{2}({T_\mu}^\alpha{}_\beta + {T_\beta}^\alpha{}_\mu + {T^\alpha}_{\mu\beta}) \, , \tag{1.5} $$

where the second expression on the right side is called the contorsion tensor. This generalization of the connection not only enables the spacetime to respond to mass as before in the general relativity, but also to spin, where spinning matter produces torsion and thus generates a non-vanishing contorsion in (1.5).

To illustrate the new features of the Einstein–Cartan theory, let us consider a Dirac spinor ψ. It is coupled to the full connection (1.5) and, especially to torsion, by means of a covariant spinor derivative. By employing the variational principle to an appropriate Lagrangian density the following field equation for the torsion is obtained [Heh 71]

$$ T_{\alpha\beta\gamma} = -\frac{1}{2} l_0^2 \, \eta_{\alpha\beta\gamma\delta} \, \overline{\psi}\gamma^5\gamma^\delta\psi \, . \tag{1.6} $$

Here $l_0^2 = \hbar c k$ is the square of the Planck length and $\eta_{\alpha\beta\gamma\delta}$ is the volume form, see (2.4). Note that the right-hand side of (1.6) is proportional to the canonical spin density of a Dirac particle, see [Rom 69, Itz 80]. Due to the presence of torsion, the energy-momentum equation, which is obtained by varying the Lagrangian density with respect to the metric, now becomes

$$ \frac{1}{k} G^*_{\alpha\beta} = T^m_{\alpha\beta} + \frac{3}{16k} l_0^4 g_{\alpha\beta} (\overline{\psi}\gamma^5\gamma^\delta\psi)(\overline{\psi}\gamma^5\gamma_\delta\psi) \, , \tag{1.7} $$

where $T^m_{\alpha\beta}$ is the usual energy-momentum tensor of Dirac particles already present in general relativity. The second term on the right side of (1.7) describes a spin-spin self-interaction induced by torsion, which was absent in the energy-momentum

[1] Besides this connection between torsion and elementary spin in the framework of general relativity, the geometrical concept of torsion is also employed in the solid state physics for the description of dislocations in solids [Bil 55, Krö 64, Krö 81, Kat 92]. Furthermore, there is an interesting link between both types of torsion-theories [Heh 65a, Heh 65b, Heh 66].

equation (1.4) of general relativity. Since this interaction occures only when matter fields overlap with each other, it is called a contact interaction. It does not only influence the curvature via (1.7), but also creates a cubic self-interaction in the Dirac equation [Heh 71]

$$i\gamma^\mu \nabla^*_\mu \psi - \frac{mc}{\hbar}\psi + \frac{3}{8} l_0^2 (\overline{\psi}\gamma^5\gamma^\delta\psi)\gamma^5\gamma_\delta\psi = 0 \;, \tag{1.8}$$

where ∇^*_μ is the covariant spinor derivative with respect to the Levi–Civita connection, see (2.35).

Besides this well-known aspect of torsion in the framework of Einstein–Cartan theory, another physical role for it has been suggested in several works on the unification of gravity and electromagnetism. These works originated from Einstein's own approach to an unified field theory [Ein 55], in which he considered an arbitrary linear connection and a non-symmetric metric $\tilde{g}_{\alpha\beta}(\neq \tilde{g}_{\beta\alpha})$, of which the antisymmetric part was identified with the dual of the electromagnetic field strength.[2] To remedy the serious drawbacks [Inf 50, Cal 53] (see also [Pau 58]) of Einstein's field theory several authors have suggested to identify the torsion trace (or torsion vector)

$$T_\mu = T^\alpha{}_{\mu\alpha} = \Gamma^\alpha{}_{\mu\alpha} - \Gamma^\alpha{}_{\alpha\mu} \tag{1.9}$$

of an arbitrary linear connection with the electromagnetic vector potential in an ad hoc manner [Bor 76a, Mof 77, Kun 79]

$$T_\mu \sim A_\mu \;, \tag{1.10}$$

still considering a non-symmetric metric. McKellar [McK 79] and Jakubiec and Kijowski [Jak 85] deduced this relation (1.10) very naturally using only the variational principle and avoiding any ad hoc assumptions. Also, the somewhat unnatural concept of a non-symmetric metric was withdrawn.

McKellar considers the usual metric and an arbitrary general linear connection $\Gamma^\alpha{}_{\mu\beta}$, which is neither compatible with the metric nor torsionless. As the result of the field equations, the connection is restricted to be of the form [McK 79]

$$\Gamma^\alpha{}_{\mu\beta} = \{^\alpha{}_{\mu\beta}\} + \frac{1}{3}\delta^\alpha{}_\beta\, T_\mu \;. \tag{1.11}$$

Furthermore, his field equations resemble precisely the source-free Einstein–Maxwell equations, provided that (1.10) is assumed.

Ferraris and Kijowski [Fer 82] arrive at the same field equations as McKellar, but they do not conclude (1.10), but consider $\Gamma^\alpha{}_{\mu\alpha} = \{^\alpha{}_{\mu\alpha}\} + \frac{4}{3}T_\mu$, which is not a

[2]Similar attempts at an unification of gravity and electromagnetism were considered by many other authors, see e.g. [Edd 21, Sch 54, Ton 55, Kur 52, Kur 74].

vector, as the electromagnetic potential and develop a U(1) gauge theory differing from the usual understanding.

Jakubiec and Kijowski consider in [Jak 85] the same theory as Ferraris and Kijowski [Fer 82], but now the relation $T_\mu \sim A_\mu$ is adopted implicitly. Although Dirac spinors are included in their unified theory [Jak 85], the employed spinor derivative is mainly built from the Levi–Civita connection, and from the general linear connection $\Gamma^\alpha{}_{\mu\beta}$, merely its trace $\Gamma^\alpha{}_{\mu\alpha}$ couples to spinors properly. Since the torsion of $\Gamma^\alpha{}_{\mu\beta}$ does not couple to spinors, the spin-torsion aspect established in Einstein–Cartan theory is missing in their theory.

The drawbacks of the above mentioned unified theories are twofold:

First, the identification $T_\mu \sim A_\mu$ (1.10) lacks a clear geometric and physical meaning, because the torsion trace is an ordinary vector but not a gauge potential. It remains invariant under U(1), while A_μ transforms in the well-known inhomogeneous way as a potential. This inconsistency can not be remedied by introducing a so-called λ–transformation, first introduced by Einstein in another context [Ein 55], by which T_μ can formally be transformed like a potential [McK 79]. What is really missing here is a clear fibre bundle geometric conception, from which a consistent U(1) theory can be deduced. Another related problem with unified field theories is the missing physical interpretation of the resulting connection (1.11): Since it is not compatible with the metric, $\nabla_\mu g_{\alpha\beta} = -\frac{2}{3}T_\mu \cdot g_{\alpha\beta} \neq 0$, it must not be applied for the parallel transports of signals on the spacetime: Otherwise, this would lead to the dependence of physical invariants upon their histories like in Weyl's unified theory [Wey 22]. Therefore, it is necessary to decompose the whole connection (1.11) into a metric part and a "non-metric" part. But this can be done in several ways, for example, as

$$\Gamma^\alpha{}_{\mu\beta} = [\{^\alpha{}_{\mu\beta}\}] + [\frac{1}{3}\delta^\alpha{}_\beta T_\mu] \quad \text{or} \tag{1.12}$$

$$\Gamma^\alpha{}_{\mu\beta} = [\{^\alpha{}_{\mu\beta}\} + \frac{1}{6}(\delta^\alpha{}_\beta T_\mu - T^\alpha g_{\mu\beta})] + [\frac{1}{6}(\delta^\alpha{}_\beta T_\mu + T^\alpha g_{\mu\beta})] \,. \tag{1.13}$$

In both examples the first bracket [...] represents a connection compatible with the metric. Although the field equations seem to suggest that the metric part of (1.11) is provided by the Christoffel symbol alone, there is no rigorous geometric justification for this assumption.

Secondly, although the linear connection used in these unifications is much more general than the Lorentzian connection (1.5) of Einstein–Cartan theory, the important spin-torsion coupling is missing either because spinning matter is not considered [McK 79, Fer 82], or because the treatment of Dirac spinors is somewhat inappropriate [Jak 85].

In my diploma thesis [Hor 94, Hor 95] I have proposed a new theory of gravity and electromagnetism, which incorporates both aforementioned aspects of torsion.

To achieve this purpose it is necessary to further expand the spacetime geometry by introducing a complex rather than a real linear connection and an *extended spinor derivative based on this connection*. Contrary to [Jak 85], this new spinor derivative not only couples the trace part, but also other components including the contorsion of the linear connection to Dirac spinors. As a consequence of this"tight" coupling, the resulting field equation for the connection can not be solved in the real numbers but require complex degrees of freedom. Thus, it is necessary to consider a complex linear connection. Through the consideration of spinorial matter it is possible to fully clarify the underlying fibre bundle geometry of this theory, and, as a consequence, especially its U(1) structure. Due to the new spinor derivative, both aspects of torsion must be revised: First, the long-standing and unsatisfactory relation (1.10) turns out to be merely a formal remnant of the new fibre bundle geometry. Instead, the electromagnetic potential A_μ is truly related to another vector part S_μ via (2.47). Secondly, the torsion-induced spin-spin contact interaction now only occurs between distinct particles. The missing of the self-interaction leads to the vanishing of the second term on the right side of (1.7) and also of the cubic spinor term in (1.8), if a one-particle system is considered.

The field equations are derived directly from the variational principle and do not require any ad hoc assumptions. Formally, they resemble precisely the well-known equations of Einstein–Maxwell theory with charged Dirac particles. But this physical interpretation is now fully justified by the structure of the underlying fibre bundle geometry, according to which the resulting complex connection can be *decomposed* into a gravitational Lorentzian (that is, compatible with the metric) connection and an electromagnetic vector potential. This splitting of the connection together with a characteristic length scale in the theory suggests that gravity and electromagnetism have the same geometrical origin.

Although the main part of this theory was developed in the diploma thesis [Hor 94], there are still many features of the theory, which were not clarified rigorously and therefore deserve detailed considerations:

First of all, the exact role of the torsion trace and its connection to the "true" underlying bundle structure were not analysed exhaustively. It was stated in [Hor 94] that the true electromagnetic vector potential is not given by the torsion but by some another vector part, S_μ, of the connection. But *formally*, the torsion trace T_μ is still related to A_μ and seems to play a role in electromagnetic phenomena. This point was not clarified in the diploma thesis. In this work I will show rigorously that torsion is connected to electromagnetism *not physically* but only *formally*. For this purpose, the electromagnetic vector potential in the resultant complex connection of the theory will be detached from the tangent frame bundle of the spacetime manifold. Since torsion is a tensor defined on the tangent bundle of the spacetime, torsion will be disconnected from electromagnetic phenomena in this way. This will also help to clarify the gauge transformation aspect of the electromagnetic potential

and its connection to torsion.

We may say that the long-standing relation $T_\mu \sim A_\mu$ is *no leading principle* for an unification of gravity and electromagnetism, but rather a *formal first hint* that both physical phenomena can be explained through the geometry of spacetime.

To understand how the vector potential originates from the intrinsic spacetime geometry, we must consider the underlying fibre bundle background geometry of our theory very carefully. This makes necessary to reconsider this fibre bundle structure developed in the diploma thesis, since there some essential points were skipped. The decomposition of the resulting complex linear connection into its metric part and an electromagnetic part, and the corresponding decomposition principle of the extended covariant spinor derivative, which are vital to the understanding of the unifying principle of our theory, will be treated in detail in this work. In so doing, we will notice why it is *not* possible in our theory to consider arbitrary U(1) principal bundles for electromagnetism but only the trivial bundle $M \times \mathrm{U}(1)$. Also, we will correct an error occured in the derivation of the connections in the diploma thesis.

In the diploma thesis, I have employed a real orthonormal tetrad field to pull back a complex connection 1-form from the complex frame bundle $F_c(M)$ onto the spacetime manifold M without further explanation. In this work I will explain and justify why this real valued structure is used in an otherwise entirely complex geometrical structure.

Finally, the spin-spin contact interaction of the new theory will now be investigated in detail by considering the energy eigenvalues of Dirac test particles in a background torsion field and also by quantizing the interaction Hamiltonian in the first Born approximation.

The organization of this work is as follows:

In the second chapter we represent the new unified field theory of gravity and electromagnetism. Although details of the computations can be found in the diploma thesis [Hor 94] and therefore will not be repeated again, the presentation in this work is kept fairly self-contained. In addition, the essential structures of the field equations are now clarified, so that they can be understood quite easily without going into details. More importantly, the physical content of the theory, which was outlined in the diploma thesis, is now explained in great detail. We clarify the basic building principle of our theory and its physical consequences. Also, the above mentioned formal aspect of torsion and its link to the basic geometrical background are explained.

In the third chapter the fibre bundle geometry of the theory is examined in every detail. First, some special topics from differential geometry are provided, which will be needed to explain the various construction steps of our fibre geometrical background: Although the basic concepts of the differential geometry like principal fibre bundle, connection 1-forms, and covariant derivatives are by now fairly well-known, there are special topics of differential geometry, which, in my opinion, are

less familiar: For example, the local representation of the fibre geometry based on cross sections, mappings of connections, and the real and complex spin geometries. After these preliminaries, the bundle geometry of our theory is constructed step by step, and the beforementioned points on the geometrical background of the theory will be discussed.

In the next chapter, we consider the spin-spin contact interaction of the new theory and discuss its differences to the interaction of the ordinary Einstein–Cartan theory. First, we study the classical Dirac equation of a test particle in a background torsion field caused by a classical plane wave field. Contrary to the contact interaction of the Einstein–Cartan theory, which is universal [Ker 75], that is, does not depend upon the interacting particle types, this is no longer the case for the new contact interaction: Now the interaction between two particles or two anti-particles differs from that between a particle and an anti-particle. However, if both types of contact interactions are quantized, and if identical particles are considered, then both interactions turn out to be non-universal.

The final chapter gives a summary of the results and an outlook on future research.

Chapter 2

The Theory of Gravity and Electromagnetism

2.1 Lagrangian density

2.1.1 Metric and tetrads

As mentioned in the introduction, the theory [Hor 94] employs a complex linear connection. Further field variables are a metric or orthonormal tetrads, and Dirac spinors. Note that tetrad fields are needed to define Dirac spinors appropriately, see e. g. [Heh 71].

To introduce these field variables and their Lagrangian density, from which the field equations will be computed using the variational principle, let us consider a real 4-dimensional spacetime manifold, denoted by M. Let $F(M)$ be its frame bundle, which is a $\mathrm{GL}(4, \mathbb{R})$ principal bundle consisting of all tangent frames. We assume that M is endowed with a pseudo-Riemannian metric $g_{\mu\nu}$, so that $F(M)$ can be reduced to the Lorentz subbundle consisting of orthonormal tangent frames only. Assuming further that M is space- and time-orientable with respect to $g_{\mu\nu}$, this Lorentz subbundle has exactly 4 connected components, by the choice of one of which we introduce a definite space- and time-orientation on M [Ble 81, Bau 81]. This chosen subbundle is called the special Lorentz bundle $L^+_\uparrow(M)$, which is a principal bundle consisting of orthonormal tangent frames such that the structure group is given by the special orthochronous Lorentz group $L^+_\uparrow$ with Lie algebra $\boldsymbol{l}$,

$$\begin{aligned} L^+_\uparrow &:= \left\{ \Lambda \in \mathrm{Mat}(4,\mathbb{R}) \,|\, \Lambda^T \eta \Lambda = \eta \,,\; \det \Lambda = 1 \,,\; \Lambda^0{}_0 \geq 1 \right\} \;; \\ \boldsymbol{l} \quad &= \left\{ \Lambda \in \mathrm{Mat}(4,\mathbb{R}) \,|\, \Lambda^T \eta + \eta \Lambda = 0 \right\} \;, \end{aligned} \tag{2.1}$$

where $\eta = (\eta_{ab}) = (\eta^{ab}) = \mathrm{diag}(1, -1, -1, -1)$. A tetrad $\sigma = (e_a{}^\mu \partial_\mu)$ is a local cross section in $L^+_\uparrow(M)$, where latin indices, running from 0 to 3, are anholonomic indices

and will be lowered and raised with η_{ab} and η^{ab}, respectively. Greek indices run also from 0 to 3 and refer to local coordinates. They are lowered and raised with $g_{\mu\nu}$ and $g^{\mu\nu}$, respectively, the latter being the inverse of $g_{\mu\nu}$. Let $(e^a{}_\mu dx^\mu)$ denote the reciprocal tetrad satisfying $e_a{}^\mu e^a{}_\nu = \delta^\mu{}_\nu$ and $e_a{}^\mu e^b{}_\mu = \delta^b{}_a$. Since the tetrad $\sigma = (e_a{}^\mu \partial_\mu)$ is orthonormal, it satisfies

$$g_{\mu\nu} e_a{}^\mu e_b{}^\nu = \eta_{ab} \;, \tag{2.2}$$

which results in the following relations:

$$g^{\mu\nu} = e_a{}^\mu e^{a\nu} \;, \quad g_{\mu\nu} = e_{a\mu} e^a{}_\nu \;, \quad g := |\det(e^a{}_\mu)| = \sqrt{-\det(g_{\mu\nu})} \;. \tag{2.3}$$

The components $e_a{}^\mu$ and $e^a{}_\mu$ will be used to convert coordinate indices to anholonomic ones and vice versa. With the help of the determinant g, the volume element on M reads

$$\eta_{\alpha\beta\gamma\delta} := g \cdot \epsilon_{\alpha\beta\gamma\delta} \;, \tag{2.4}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is totally antisymmetric in its indices and $\epsilon_{0123} = +1$.

2.1.2 Complex linear connection

Let $\mathbb{C} \otimes TM$ be the complexified tangent bundle of M. Contrary to $F(M)$, the complex frame bundle $F_c(M)$ consists of all complex tangent frames of $\mathbb{C} \otimes TM$ and is a $\mathrm{GL}(4, \mathbb{C})$ principal bundle naturally containing $F(M)$ as a canonical subbundle. Since $L^+_\uparrow(M)$ is naturally contained in $F(M)$, a tetrad $\sigma = (e_a{}^\mu \partial_\mu)$ is in particular a cross section in $F(M)$ and thus also in $F_c(M)$.[1]

A *complex linear connection* ω is a $\mathrm{GL}(4, \mathbb{C})$ connection on $F_c(M)$, which can be pulled back to M locally with the tetrad σ, yielding a $\boldsymbol{gl}(4,\mathbb{C})$-valued 1-form

$$(\sigma^*\omega)^a{}_b =: \Gamma^a{}_{\mu b}\, \mathrm{d}x^\mu \;, \tag{2.5}$$

which we call also a complex linear connection.

In the first instance, the real-valuedness of the tetrad σ seems to be confusing with respect to the complex structures introduced. We remark that the whole theory remains valid if we allow also for complex tetrads, which are cross sections of the special complex Lorentz bundle $\mathbb{C}L^+(M)$ containing not only real, but also all complex orthonormal tangent frames in $\mathbb{C} \otimes TM$, see 3.2.4. This is due to the fact that the metric, and hence all expressions derived from it (like the Levi–Civita connection, Einstein–tensor etc.) and also the matter currents $\overline{\psi}\gamma^\mu\psi$ and $\overline{\psi}\gamma^\mu\gamma^5\psi$ are gauge-invariant expressions and thus remain real valued in any case, guaranteeing the same field equations and their physical interpretations as in the real tetrad case.

[1] For more information on differential geometry see the next chapter.

In order to show this invariance explicitly, let $(f_a{}^\mu \partial_\mu)$ denote a complex tetrad field. Then, since any two tetrads, regarded as cross sections into the complex Lorentz bundle $\mathbb{C}L^+(M)$, are connected by a gauge transformation (see 3.1.6) of the complex Lorentz group $\mathbb{C}L^+$, there exists a $\mathbb{C}L^+$-valued function $\Lambda^b{}_a$ with

$$f_a{}^\mu = e_b{}^\mu \Lambda^b{}_a \ , \tag{2.6}$$

where $e_b{}^\mu$ are the components of a real tetrad, which is viewed here as a special complex tetrad. Since, by definition of $\mathbb{C}L^+$ (see (3.59) or simply the complexified version of (2.1)),

$$\Lambda^c{}_a \eta_{cd} \Lambda^d{}_b = \eta_{ab} \ , \tag{2.7}$$

we have for the inverse metric the desired invariance:

$$g^{\mu\nu} = f_a{}^\mu \eta_{ab} f_b{}^\nu = e_a{}^\mu \eta_{ab} e_b{}^\nu \ . \tag{2.8}$$

Thus in particular, the metric $g^{\mu\nu}$ and also $g_{\mu\nu}$ are always real valued quantities. The real valuedness of the currents $\overline{\psi}\gamma^\mu\psi$ and $\overline{\psi}\gamma^5\gamma^\mu\psi$ can be verified in a similar fashion, using a spin gauge transformation instead of the Lorentz transformation.

The reason why we have restricted the tetrads to be real valued is that we want to avoid confusion concerning their physical meaning as orthonormal reference frames, this being necessary for example to describe the physics studied in an observer's laboratory [Mis 73].

The connection was introduced in (2.5) in its anholonomic, tetrad components. Its corresponding coordinate components are obtained by the rule

$$\Gamma^\alpha{}_{\mu\beta} = e_a{}^\alpha \, e^b{}_\beta \Gamma^a{}_{\mu b} + e_c{}^\alpha \partial_\mu e^c{}_\beta \ . \tag{2.9}$$

These components transform in the well-known inhomogeneous way under coordinate changes. The curvature tensor, Ricci tensor, curvature scalar, and the curvature trace of the complex linear connection are defined in its anholonomic components as follows[2]

$$\begin{aligned}
R^a{}_{b\mu\nu} &= \partial_\mu \Gamma^a{}_{\nu b} + \Gamma^a{}_{\mu c}\Gamma^c{}_{\nu b} - \partial_\nu \Gamma^a{}_{\mu b} - \Gamma^a{}_{\nu c}\Gamma^c{}_{\mu b} \ ; &\text{(2.10a)}\\
R_{\mu\nu} &= R^a{}_{b\alpha\nu} \cdot e_a{}^\alpha \, e^b{}_\mu \ ; &\text{(2.10b)}\\
R &= R^a{}_{b\mu\nu} \cdot e_a{}^\mu \, e^{b\nu} \ ; &\text{(2.10c)}\\
Y_{\mu\nu} &= R^a{}_{a\mu\nu} = \partial_\mu \Gamma^a{}_{\nu a} - \partial_\nu \Gamma^a{}_{\mu a} \ . &\text{(2.10d)}
\end{aligned}$$

It is important to note that since $\Gamma^a{}_{\mu b}$ is an arbitrary complex linear connection, it is not compatible with the metric in general. Using the above coordinate expression

[2] We remark that the holonomic, i.e. coordinate, components of the various tensor quantities in (2.10) can be obtained simply by employing the tetrad components. For example, $R^\alpha{}_{\beta\mu\nu} = R^a{}_{b\mu\nu} e_a{}^\alpha e^b{}_\beta$.

(2.9) it is easy to show the following equivalence

$$\begin{aligned}\nabla_\mu g_{\alpha\beta} &= \partial_\mu g_{\alpha\beta} - \Gamma^\epsilon{}_{\mu\alpha} g_{\epsilon\beta} - \Gamma^\epsilon{}_{\mu\beta} g_{\alpha\epsilon} \;=\; \partial_\mu g_{\alpha\beta} - \Gamma_{\beta\mu\alpha} - \Gamma_{\alpha\mu\beta} \\ &= \partial_\mu (e^c{}_\alpha e_{c\beta}) - (e^b{}_\beta e^a{}_\alpha \Gamma_{b\mu a} + e_{c\beta}\partial_\mu e^c{}_\alpha + e^a{}_\alpha e^b{}_\beta \Gamma_{a\mu b} + e_{c\alpha}\partial_\mu e^c{}_\beta) \\ &= -e^a{}_\alpha e^b{}_\beta (\Gamma_{b\mu a} + \Gamma_{a\mu b}) \;.\end{aligned}$$

Thus we obtain a simple *metricity condition* in terms of the anholonomic connection components,

$$\nabla_\mu g_{\alpha\beta} = 0 \quad \Leftrightarrow \quad \Gamma_{a\mu b} + \Gamma_{b\mu a} = 0 \;, \tag{2.11}$$

where the right equation is precisely the condition of the Lie algebra $\boldsymbol{l}$ (2.1) of the Lorentz group $L^+_\uparrow$ (to be more precise, of its complexified version), see (3.48). Therefore, a connection which is compatible with the metric will be called henceforth a *Lorentzian connection.*

Since in our theory $\Gamma^a{}_{\mu b}$ does not satisfy (2.11), its trace $\Gamma^a{}_{\mu a}$ does not vanish in (2.10d) in general. Note that $\Gamma^a{}_{\nu a}$ is a vector, contrary to $\Gamma^\alpha{}_{\nu\alpha}$.

2.1.3 Extended spinor derivative

It is well-known that spinor derivatives can be constructed not only from the Christoffel symbol (see, for example, [DeW 64]) but also from any Lorentzian connection with non-vanishing contorsion [Heh 71]. Such a connection $\Gamma^a{}_{\mu b}$ compatible with the metric defines the following spinor derivative (for details, see 3.2.3)

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{4}\Gamma_{a\mu b}\gamma^b\gamma^a\psi \;, \tag{2.12}$$

where the γ–matrices satisfy $\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}\mathbb{1}$. In (2.12) the product $\gamma^b\gamma^a$ has been employed instead of the commonly used Lorentz generators[3]

$$\sigma^{ba} = \frac{1}{2}(\gamma^b\gamma^a - \gamma^a\gamma^b) \tag{2.13}$$

in virtue of the metricity condition $\Gamma_{a\mu b} = -\Gamma_{b\mu a}$ [Heh 71, Heh 91, Law 89, Ber 91]. If we now omit this condition and use our complex linear connection instead, its non-vanishing trace part $\Gamma_{a\mu b}\cdot \frac{1}{2}(\gamma^b\gamma^a + \gamma^a\gamma^b) = \Gamma_{a\mu b}\eta^{ba} = \Gamma^a{}_{\mu a}$ also contributes to the spinor derivative,

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{4}\Gamma_{a\mu b}\gamma^b\gamma^a\psi = \partial_\mu\psi - \frac{1}{4}\Gamma_{a\mu b}\sigma^{ba}\psi - \frac{1}{4}\Gamma^a{}_{\mu a}\psi \;. \tag{2.14}$$

Note that at this stage (2.14) is rather a formal extension since it is only $L^+_\uparrow$ covariant but of course not with respect to GL(4, $\mathbb{C}$). Its full geometric meaning is expounded

[3]In physics, Lorentz generators are usually defined to be i times our σ^{ba}, which then are hermitian matrices. But for our purposes it is more convenient to work without the factor i.

in the next chapter. We remark that this extension is not unique, since σ^{ba} could have been replaced equally well by $-\gamma^a\gamma^b$ or, more generally, by $\sigma^{ba}+\varepsilon\cdot\eta^{ba}$. Due to this freedom, spinors with any multiple of the elementary charge, εe, can be treated, see 2.4.

2.1.4 Lagrangian density

We introduce the adjoint spinor $\overline{\psi} := \psi^\dagger\gamma^0$, $\gamma^\mu := \gamma^a\, e_a{}^\mu$, the mass of the spinor particle m, $k = 8\pi G/c^4$, the Planck length $l_0 := \sqrt{\hbar c k} \approx 8\cdot 10^{-35}$m and a length scale l to be determined later. Using the extended spinor derivative and the curvature expressions derived from the complex linear connection in (2.10) we write down the following Lagrangian density

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_m + \mathcal{L}_G + \mathcal{L}_Y &\text{(2.15a)}\\
&=: g\cdot\hbar c\left[i\overline{\psi}\gamma^\mu\nabla_\mu\psi - \frac{mc}{\hbar}\overline{\psi}\psi\right] - \frac{g}{2k}R + \frac{g}{4k}l^2 Y_{\mu\nu}Y^{\mu\nu} &\text{(2.15b)}\\
&= \frac{g}{k}\cdot\left[il_0^2\,\overline{\psi}\gamma^\mu\nabla_\mu\psi - mc^2k\overline{\psi}\psi - \frac{1}{2}R + \frac{1}{4}l^2 Y_{\mu\nu}Y^{\mu\nu}\right]. &\text{(2.15c)}
\end{aligned}
$$

Note that this Lagrangian is complex valued. We must consider this whole complex expression and not only its real part, since otherwise the contribution of the full complex connection would be taken away from the theory, making it meaningless.[4] Obviously, the three parts $\mathcal{L}_m$, $\mathcal{L}_G$, and $\mathcal{L}_Y$ resemble the usual Lagrangian densities of spinorial matter, gravity, and the electromagnetic field, respectively. But now they are all complex valued and, whereas expressions similar to $\mathcal{L}_G$ and $\mathcal{L}_Y$ for a real connection were already used in [McK 79, Jak 85], the matter Lagrangian $\mathcal{L}_m$ including the extended spinor derivative (2.14) is new and plays a key role in our unification.

Note also that the introduction of the characteristic length l is necessary from purely dimensional arguments.[5] In (2.15) it is possible to relate this length scale l with the already given Planck length l_0. Regarding the last term of (2.15c), we recognize l^2 as the self-coupling constant of the connection and, looking at the first term, l_0^2 as the coupling strength between the connection and the Dirac particles. Since Dirac fields are vector fields on a certain spinor bundle, which in turn is closely related to the intrinsic spacetime structure,[6] it is legitimate to consider Dirac spinors as intrinsical geometrical objects of the spacetime, at least on the non-quantum

[4] For another interesting complex Lagrangian theory of gravity we refer to Ashtekar's formulation of general relativity [Ash 91], which might be related to the complex structures developed in this chapter, cf. [Mag 87, Gam 93].

[5] The partial derivatives ∂_μ and the connection have the dimension of inverse length.

[6] Spinors can be introduced on a spacetime manifold M provided that it possesses a spin-structure with respect to a given metric, see 3.2. The existence of such a structure is a topological property of M [Bau 81, Law 89]. Since in most cases, a spacetime M posseses a spin structure

level. Thus, l_0 and l both describe couplings between objects belonging to the same intrinsic spacetime geometry and should therefore be of the same order of magnitude. If, on the other hand, the unknown length l turns out to be drastically different from the Planck length l_0, we may say that the Lagrangian (2.15) does not provide a physically sensible theory.

2.2 Field equations

The field equations are obtained by the action principle based on the Lagrangian constructed in (2.15). The variation acts independently on the field quantities $\Gamma^a{}_{\mu b}$, ψ, $\overline{\psi}$ and $e_a{}^\mu$. Since in our Lagrangian no second derivative is present, the Euler–Lagrange equations will be of the simple form

$$0 = \frac{\delta \mathcal{L}}{\delta v} := \frac{\partial \mathcal{L}}{\partial v} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\nu v} \, , \tag{2.16}$$

where v is an arbitrary field component. In the following only an outline of the calculations is presented. For the full computations we refer to [Hor 94].

2.2.1 Field equation for the connection

To simplify the computations, we subtract the Levi–Civita connection (1.3) from the complex connection (in its coordinate components, see (2.9))

$$\Sigma^\alpha{}_{\mu\beta} := \Gamma^\alpha{}_{\mu\beta} - \{^{\alpha}{}_{\mu\beta}\} \, . \tag{2.17}$$

Being the difference of two linear connections, $\Sigma^\alpha{}_{\mu\beta}$ is a third rank tensor, which is complex valued in general. As mentioned in [Hor 94], any such third rank tensor admits a "4-vector decomposition"

$$\Sigma_{\alpha\beta\gamma} = Q_\alpha g_{\beta\gamma} + S_\beta g_{\alpha\gamma} + g_{\alpha\beta} U_\gamma - \frac{1}{12} \eta_{\alpha\beta\gamma\delta} V^\delta + \Upsilon_{\alpha\beta\gamma} \, , \tag{2.18}$$

where the four vectors Q_α, S_β, U_γ, V_δ and the "tensor rest" $\Upsilon_{\alpha\beta\gamma}$ are defined in the appendix, see (A.2). This tensor rest[7] satisfies $\Upsilon^\alpha{}_{\alpha\gamma} = \Upsilon^\alpha{}_{\gamma\alpha} = \Upsilon_\gamma{}^\alpha{}_\alpha = \Upsilon_{[\alpha\beta\gamma]} = 0$. The volume element $\eta_{\alpha\beta\gamma\delta}$ was defined in (2.4). With (2.17) and (2.18) the connection can be decomposed as[8]

$$\Gamma^\alpha{}_{\mu\beta} = \{^{\alpha}{}_{\mu\beta}\} + Q^\alpha g_{\mu\beta} + S_\mu \delta^\alpha{}_\beta + \delta^\alpha{}_\mu U_\beta - \frac{1}{12} \eta^\alpha{}_{\mu\beta\gamma} V^\delta + \Upsilon^\alpha{}_{\mu\beta} \, . \tag{2.19}$$

[Ger 68, Ger 70], we may say that Dirac spinors are natural geometric objects on M like ordinary vector fields on the tangent bundle.

[7] For an explicit example of $\Upsilon_{\alpha\beta\gamma}$ see (A.5).

[8] Note that this decomposition is not irreducible in the sense of [McC 92], but is introduced for computational convenience.

Using (2.17) only, the field equation for $\Gamma^a{}_{\mu b}$ following from (2.15) reads

$$0 = \frac{\delta}{\delta \Gamma^a{}_{\mu b}} \left(\mathcal{L}_m + \mathcal{L}_G + \mathcal{L}_Y\right) \cdot \delta^\gamma{}_\mu \, e^{a\alpha} \, e_b{}^\beta \cdot \frac{k}{g} \qquad \Leftrightarrow \tag{2.20}$$

$$0 = -\frac{1}{4} i l_0^2 \, \overline{\psi} \gamma^\gamma \gamma^\beta \gamma^\alpha \psi - \frac{1}{2} \left[\Sigma^{\beta\epsilon}{}_\epsilon g^{\alpha\gamma} + \Sigma^\epsilon{}_\epsilon{}^\alpha g^{\gamma\beta} - \Sigma^{\beta\alpha\gamma} - \Sigma^{\gamma\beta\alpha}\right] - l^2 g^{\alpha\beta} \, \nabla^*_\nu Y^{\nu\gamma} \ , \tag{2.21}$$

where ∇^*_ν denotes the covariant differentiation with respect to $\{^\alpha{}_{\mu\beta}\}$. The bracket $[\ldots]$ on the left-hand side pertains to the Lagrangian $\mathcal{L}_G$ and can be expressed with the 4-vector decomposition (2.18) as follows:

$$\begin{aligned} -\frac{1}{2}[\ldots] \;\; &= \;\; -\frac{1}{2}[(Q^\alpha + 3U^\alpha) g^{\beta\gamma} + (U^\beta + 3Q^\beta) g^{\alpha\gamma} - (Q^\gamma + U^\gamma) g^{\alpha\beta} \\ &\qquad + \frac{1}{6} \eta^{\gamma\beta\alpha\delta} V_\delta - (\Upsilon^{\beta\alpha\gamma} + \Upsilon^{\gamma\beta\alpha})] \ . \end{aligned} \tag{2.22}$$

Note that in (2.22) there is no term containing derivatives.[9] We now define the vector current $j^\alpha := \overline{\psi} \gamma^\alpha \psi$ and the axial current $j^{5\,\delta} := \overline{\psi} \gamma^5 \gamma^\delta \psi$. Inserting (2.22) into (2.21) and contracting (2.21) successively with $g_{\beta\gamma}$, $g_{\alpha\gamma}$, $g_{\alpha\beta}$ and $1/6 \cdot \eta_{\gamma\beta\alpha\delta}$ we obtain in this order

$$\begin{aligned} 0 &= -i l_0^2 \cdot j^\alpha - 3Q^\alpha - 6U^\alpha - l^2 \nabla^*_\nu Y^{\nu\alpha} && (2.23a) \\ 0 &= \frac{1}{2} i l_0^2 \cdot j^\beta - 6Q^\beta - 3U^\beta - l^2 \nabla^*_\nu Y^{\nu\beta} && (2.23b) \\ 0 &= -i l_0^2 \cdot j^\gamma - 4 l^2 \, \nabla^*_\nu Y^{\nu\gamma} && (2.23c) \\ 0 &= -\frac{1}{4} l_0^2 \cdot j^5{}_\delta + \frac{1}{12} V_\delta && (2.23d) \end{aligned}$$

One can easily derive

$$-\,Q^\alpha = U^\alpha = -i l_0^2 / 4 \cdot j^\alpha \ , \qquad V_\delta = 3 l_0^2 j^5_\delta \ . \tag{2.24}$$

Inserting this result and (2.23c) back into (2.21) we obtain $\Upsilon_{\alpha\beta\gamma} = 0$. Since the Levi–Civita connection is compatible with the metric, it follows from (2.11) that its anholonomic components fulfill $\{^a{}_{\mu a}\} = 0$. Using this fact and the equations (2.17) and (2.24), it follows

$$\Gamma^a{}_{\mu a} = \Sigma^a{}_{\mu a} = Q_\mu + 4S_\mu + U_\mu = 4S_\mu \tag{2.25}$$

[9] This pleasant feature is due to the decomposition (2.17): In calculating $\delta\mathcal{L}_G/\delta\Gamma^a{}_{\mu b}$, we encounter the expression $\partial_\nu(\delta\mathcal{L}_G/\partial_\nu\delta\Gamma^a{}_{\mu b})$ according to (2.16). As can be seen from the structure of the curvature (2.10), this expression contains merely partial derivatives of various tetrad components. But these cancel exactly with the Levi–Civita parts contained in $\partial\mathcal{L}_G/\partial\Gamma^a{}_{\mu b}$. For details we refer to [Hor 94].

and thus from (2.10d)

$$Y_{\mu\nu} = 4S_{\mu\nu} := 4(\partial_\mu S_\nu - \partial_\nu S_\mu) \; . \tag{2.26}$$

Inserting (2.24) and $\Upsilon_{\alpha\beta\gamma} = 0$ into (2.19) we obtain

$$\begin{aligned} \Gamma^a{}_{\mu b} &= \widehat{\Gamma}^a{}_{\mu b} + \delta^a{}_b \cdot S_\mu \; , \qquad \text{where} && (2.27)\\ \widehat{\Gamma}^a{}_{\mu b} &:= \{^a{}_{\mu b}\} + \frac{1}{4} l_0^2 \left(i \cdot j^a e_{b\mu} - i \cdot e^a{}_\mu j_b - \eta^a{}_{\mu b d} j^{5\,d} \right) && (2.28) \end{aligned}$$

and, from (2.23c),

$$16i(l^2/l_0^2)\, \nabla^*_\nu S^{\nu\gamma} = j^\gamma \; . \tag{2.29}$$

The last equation implies the current conservation

$$\nabla^*_\gamma j^\gamma = 16i(l^2/l_0^2)\, \nabla^*_\gamma \nabla^*_\nu S^{\nu\gamma} = 0 \; . \tag{2.30}$$

Note that so far we have not used the complex extension of the connection explicitly in the calculations. But from (2.27), (2.28) and (2.29) it follows that the connection must be complex valued. In other words, these equations can not be solved using a real connection only. This is exactly the reason why we have chosen a complex rather than a real linear connection as our field variable. It is obvious that these complex contributions are solely due to the presence of Dirac fields, since, if Dirac fields were not present, all of the above field equations could be considered real valued: In this case, we would obtain instead of (2.27) and (2.29)

$$\begin{aligned} \Gamma^a{}_{\mu b} &= \{^a{}_{\mu b}\} + \delta^a{}_b \cdot S_\mu \; , && (2.31a)\\ 0 &= 16i(l^2/l_0^2)\, \nabla^*_\nu S^{\nu\gamma} \; . && (2.31b) \end{aligned}$$

Thus, without Dirac spinors, complex geometric structures become superfluous.

We observe, that the connection part $\widehat{\Gamma}^a{}_{\mu b}$ satisfies

$$\widehat{\Gamma}_{a\mu b} = -\widehat{\Gamma}_{b\mu a} \tag{2.32}$$

and therefore defines a *Lorentzian* connection, that is, a connection compatible with the metric, cf. (2.11).

2.2.2 Dirac equations

The Lagrangian (2.15) immediately yields $0 = \delta\mathcal{L}/\delta\overline{\psi} = \partial\mathcal{L}_m/\partial\overline{\psi}$ or, equivalently,

$$i\gamma^\mu \nabla_\mu \psi - \frac{mc}{\hbar}\psi = 0 \; , \tag{2.33}$$

wherein ∇_μ is given by the extended covariant spinor derivative (2.14). The equation (2.33) can be reexpressed with the help of (2.14) and (2.18) as follows:

$$i\gamma^\mu(\nabla^*_\mu - S_\mu)\psi - \frac{mc}{\hbar}\psi + \left(\frac{3}{2}iU_\mu + \frac{1}{8}V_\mu\gamma^5\right)\gamma^\mu\psi = 0 \, , \tag{2.34}$$

where the symbol ∇^*_μ is used for the covariant derivative with respect to $\{^\alpha{}_{\mu\beta}\}$ as well as for the corresponding spinor derivative

$$\nabla^*_\mu\psi := \partial_\mu\psi - \frac{1}{4}\{_{a\mu b}\}\gamma^b\gamma^a\psi \, . \tag{2.35}$$

Upon inserting the field equation for the connection (2.24), replacing the vectors U_μ and V_μ by $U_\mu = -il_0^2/4\, j_\mu$ and $V_\mu = 3l_0^2 j^5_\mu$, respectively, the Dirac equation becomes

$$i\gamma^\mu(\nabla^*_\mu - S_\mu)\psi - \frac{mc}{\hbar}\psi + \frac{3}{8}l_0^2(j_\mu + j^5{}_\mu\gamma^5)\gamma^\mu\psi = 0 \, . \tag{2.36}$$

The spinor equation for $\overline{\psi}$ is more difficult to compute [Hor 94]. The result is

$$i(\nabla^*_\mu + S_\mu)\overline{\psi}\cdot\gamma^\mu + \frac{mc}{\hbar}\overline{\psi} - \frac{3}{8}l_0^2\overline{\psi}(j_\mu + j^5{}_\mu\gamma^5)\gamma^\mu = 0 \tag{2.37}$$

with $\nabla^*_\mu\overline{\psi} = \overline{\nabla^*_\mu\psi}$. Contrary to the Dirac equation of the Einstein–Cartan theory (1.8), the nonlinear terms in (2.36) and (2.37) vanish due to the identity

$$(j_\mu + j^5{}_\mu\gamma^5)\gamma^\mu\psi = (\overline{\psi}\gamma_\mu\psi + \overline{\psi}\gamma^5\gamma_\mu\psi\,\gamma^5)\gamma^\mu\psi = 0 \, , \tag{2.38}$$

see [Hor 94]. Note that (2.38) is more stringent than the well-known Pauli relation $j^\mu j_\mu + j^{5\mu}j^5{}_\mu = 0$, which can be obtained from (2.38) by contracting from left by $\overline{\psi}$.

Since (2.37) is the spinor equation for the adjoint spinor $\overline{\psi}$, it must agree with the adjoint of the first equation (2.36). This immediately implies that S_μ is purely imaginary,

$$\mathrm{Re}(S_\mu) = 0 \, . \tag{2.39}$$

2.2.3 Field equation for the tetrad

The Lagrangian (2.15) contains no derivatives of $e_a{}^\mu$. With the help of (2.3) we get

$$\begin{aligned} 0 &= \frac{\delta\mathcal{L}}{\delta e_c{}^\alpha}e_{c\beta} = \frac{\partial}{\partial e_c{}^\alpha}\left(\mathcal{L}_m + \mathcal{L}_G + \mathcal{L}_Y\right)\cdot e_{c\beta} \\ &= \left[-\mathcal{L}_m\, g_{\alpha\beta} + g\, i\hbar c\overline{\psi}\gamma_\alpha\nabla_\beta\psi\right] - \frac{1}{2k}\left[-gR\, g_{\alpha\beta} + gR_\alpha{}^\mu{}_{\beta\mu} + gR^\mu{}_{\alpha\mu\beta}\right] \\ &\quad + \frac{1}{4k}l^2\left[-gY_{\mu\nu}Y^{\mu\nu}g_{\alpha\beta} + 4gY_{\mu\alpha}Y^\mu{}_\beta\right] \, . \end{aligned} \tag{2.40}$$

In order to elaborate the physical content of this equation, we insert the field equations for the connection (2.27), (2.28), (2.29) and the Dirac equations (2.36) and (2.37) together with (2.38) and obtain the following equation, where the brackets [...] correspond to respective brackets in (2.40)

$$\begin{aligned} 0 \;=\; & g[i\hbar c\overline{\psi}\gamma_\alpha(\nabla^*_\beta - S_\beta)\psi - \frac{1}{8}\hbar c l_0^2(j_\alpha j_\beta + j^5_\alpha j^5_\beta)] \\ & -\frac{g}{2k}[2G^*_{\alpha\beta} + i\hbar ck(\overline{\psi}\gamma_\alpha\nabla^*_\beta\psi + \nabla^*_\beta\overline{\psi}\cdot\gamma_\alpha\psi - \frac{1}{2}\nabla^{*\gamma}(\overline{\psi}\gamma_{[\alpha}\gamma_\beta\gamma_{\gamma]}\psi)) \\ & \qquad - \frac{1}{4}i\hbar ckl_0^2(j_\alpha j_\beta + j^5_\alpha j^5_\beta)] \\ & +\frac{g}{4k}l^2[64S_{\alpha\gamma}S_\beta{}^\gamma - 16S_{\mu\nu}S^{\mu\nu}g_{\alpha\beta}]\;. \end{aligned} \tag{2.41}$$

In the second bracket, the Einstein-tensor

$$G^*_{\alpha\beta} = R^*_{\alpha\beta} - \frac{1}{2}R^* g_{\alpha\beta} \tag{2.42}$$

is built from the Ricci tensor $R^*_{\alpha\beta}$ and the Ricci scalar R^* of the Levi–Civita connection. Note that the current-current term $(j_\alpha j_\beta + j^5_\alpha j^5_\beta)$ in the first bracket comes from the spinor derivative, whereas the corresponding expression in the second bracket is the result of the variation of the curvature scalar. Both current-current terms cancel each other and we can reexpress (2.41) as follows

$$T^G_{\alpha\beta} \;=\; T^m_{\alpha\beta} + T^S_{\alpha\beta}\,, \quad \text{where} \tag{2.43a}$$

$$T^G_{\alpha\beta} \;:=\; \frac{1}{k}\left(R^*_{\alpha\beta} - \frac{1}{2}R^* g_{\alpha\beta}\right)\;; \tag{2.43b}$$

$$T^m_{\alpha\beta} \;:=\; \frac{i\hbar c}{2}\left[\overline{\psi}\gamma_\alpha(\nabla^*_\beta - S_\beta)\psi - (\nabla^*_\beta + S_\beta)\overline{\psi}\cdot\gamma_\alpha\psi + \frac{1}{2}\nabla^{*\gamma}(\overline{\psi}\gamma_{[\alpha}\gamma_\beta\gamma_{\gamma]}\psi)\right]\;; \tag{2.43c}$$

$$T^S_{\alpha\beta} \;:=\; \frac{16}{k}l^2\left[S_{\alpha\gamma}S_\beta{}^\gamma - \frac{1}{4}S_{\mu\nu}S^{\mu\nu}g_{\alpha\beta}\right]\;. \tag{2.43d}$$

Since $T^G_{\alpha\beta}$ and $T^S_{\alpha\beta}$ are symmetric in α and β, $T^m_{\alpha\beta}$ has this property too, due to (2.43a). Indeed, a lengthy calculation [Hor 94] gives

$$T^m_{\alpha\beta} = \frac{i\hbar c}{4}\left[\overline{\psi}\gamma_\alpha(\nabla^*_\beta - S_\beta)\psi - (\nabla^*_\beta + S_\beta)\overline{\psi}\cdot\gamma_\alpha\psi + (\alpha\leftrightarrow\beta)\right]\;. \tag{2.44}$$

Since $T^G_{\alpha\beta}$ is proportional to the Einstein–tensor, we obtain the conservation law

$$0 = \nabla^*_\alpha(T^G)^{\alpha\beta} = \nabla^*_\alpha(T^m)^{\alpha\beta} + \nabla^*_\alpha(T^S)^{\alpha\beta}\;. \tag{2.45}$$

2.3 Physical interpretation

2.3.1 Formal aspects of gravity and electromagnetism

The field equations (2.36), (2.37), (2.29) and (2.43),

$$i\gamma^\mu(\nabla^*_\mu - S_\mu)\psi - \frac{mc}{\hbar}\psi = 0 \; ; \tag{2.46a}$$

$$i(\nabla^*_\mu + S_\mu)\overline{\psi}\cdot\gamma^\mu + \frac{mc}{\hbar}\overline{\psi} = 0 \; ; \tag{2.46b}$$

$$16i(l^2/l_0^2)\,\nabla^*_\nu S^{\nu\gamma} = j^\gamma \; ; \tag{2.46c}$$

$$T^G_{\alpha\beta} = T^m_{\alpha\beta} + T^S_{\alpha\beta} \; , \tag{2.46d}$$

resemble the well-known structures of the Einstein–Maxwell theory, provided that the vector S_μ is identified with the electromagnetic potential A_μ. But the factor of proportionality between S_μ and A_μ remains undetermined because the charge of the Dirac particle is not fixed. However, the gauge property (2.52) below shows that this charge has to be negative, cf. [Itz 80]. We therefore identify ψ with electron carrying the negative elementary charge $-e$, and we make the following identification

$$S_\mu = \frac{ie}{\hbar c}A_\mu \; . \tag{2.47}$$

Since here the vector S_μ is purely imaginary, this identification is in accordance with (2.39). With (2.47) the equation (2.46a) describes the Dirac equation for an electron in a curved spacetime, (2.46b) being its adjoint. In order to accomodate (2.46c) exactly to the inhomogeneous Maxwell equation in the curved spacetime of general relativity (cf. [Mis 73])

$$-\,ej^\gamma = \nabla^*_\nu F^{\nu\gamma} \; , \tag{2.48}$$

we adjust the length scale l

$$j^\gamma = 16i(l^2/l_0^2)\,\nabla^*_\nu S^{\nu\gamma} = 16i(l^2/l_0^2)\,\frac{ie}{\hbar c}\,\nabla^*_\nu F^{\nu\gamma} \stackrel{!}{=} \frac{1}{-e}\,\nabla^*_\nu F^{\nu\gamma} \qquad \Leftrightarrow$$
$$l^2 = \frac{1}{64\pi}l_0^2\frac{\hbar c}{e^2/4\pi} = \frac{1}{64\pi\,\alpha}\,l_0^2 \quad \Rightarrow l \approx 0.83\,l_0 \; , \tag{2.49}$$

where α is the fine structure constant and where we have employed Heaviside–Lorentz units, cf. [Jac 65]. Inserting this result into (2.43d) the last equation (2.46d) becomes the energy-momentum equation of general relativity including the energy-momentum tensors of gravity $T^G_{\alpha\beta}$ (2.43b), of electron $T^m_{\alpha\beta}$ (2.43c), and of the electromagnetic field $T^S_{\alpha\beta}$ (2.43d). Moreover, if (2.47), (2.49), and the field equations (2.27), (2.28), (2.29) for the connection are inserted back into the Lagrangian (2.15), this Lagrangian becomes the familiar Einstein–Maxwell Lagrangian

$$\mathcal{L} = g\cdot\hbar c\left[i\overline{\psi}\gamma^\mu(\nabla^*_\mu - \frac{ie}{\hbar c}A_\mu)\psi - \frac{mc}{\hbar}\overline{\psi}\psi\right] - \frac{g}{2k}\,R^* - \frac{g}{4}\,F_{\mu\nu}F^{\mu\nu} \; . \tag{2.50}$$

2.3.2 Geometric interpretation of electromagnetism

We have shown that *formally*, the field equations of our theory completely agree with those of Einstein–Maxwell theory. But there are important differences concerning the physical understanding of the electromagnetism: Contrary to the ordinary theory *our interpretation of electromagnetism is geometric.* To explain this view, we briefly discuss the fibre bundle structure expounded in the next chapter and give rigorous geometric meanings to the field equation (2.27) for the connection and to the identification (2.47).

We first remark that the whole connection (2.27) is not a Lorentzian connection, since it is not compatible with the metric, that is, its covariant derivative of the metric does not vanish:

$$\begin{aligned} \nabla_\mu g_{\alpha\beta} &= \partial_\mu g_{\alpha\beta} - (\widehat{\Gamma}^\epsilon{}_{\mu\alpha} + \delta^\epsilon{}_\alpha S_\mu) g_{\epsilon\beta} - (\widehat{\Gamma}^\epsilon{}_{\mu\beta} + \delta^\epsilon{}_\beta S_\mu) g_{\alpha\epsilon} \qquad \Leftrightarrow \\ \nabla_\mu g_{\alpha\beta} &= -2S_\mu \cdot g_{\alpha\beta} \neq 0 \ , \end{aligned} \tag{2.51}$$

where we have used the fact that $\widehat{\Gamma}^a{}_{\mu b}$ in (2.28) defines a Lorentzian connection compatible with the metric, see (2.32). From the equation (2.51) we conclude that the vector S_μ, which was identified with the electromagnetic potential via (2.47), is a non-metricity vector, cf. [McC 92]. Thus we can say that electromagnetism is related to non-metricity rather than to torsion. But this hasty conclusion must be regarded with caution, since the fibre bundle geometry of our theory interprets S_μ as something completely different, and, in this setting, S_μ is really a true U(1) potential and nothing else, as the next chapter and also the discussions below will show.

The fact that the resultant connection in (2.27) is not compatible with the metric means that our theory can not be immersed into a Riemann–Cartan geometry, where the connection is assumed to be metric compatible from the outset, but that it requires the full GL(4, $\mathbb{C}$) structure. A similar statement also holds for the unified field theories mentioned in the introduction, since in these theories the geometry requires the full GL(4, $\mathbb{R}$) structure of the real frame bundle $F(M)$. As mentioned thereby, one of the problems with unified field theories is the lack of an unique geometric prescription of how to separate the resultant connection (1.11).

In principle, it not difficult to provide such a decomposition prescription, which will be developed in detail in the next chapter. Here we shall explain the main idea of this decomposition principle: The starting point is the well-known fact, that a general linear connection is represented by a connection 1-form ω (see 3.1.7) on the tangent frame bundle $F(M)$ of the spacetime manifold M. Suppose now that ω can definitely be mapped (to be more precise, be pulled back) to a connection 1-form defined on a certain *fibre-product bundle* (see 3.1.3) built of the special Lorentz bundle $L^+_\uparrow(M)$ (2.1) and some yet unknown U(1) bundle U(1)(M). As explained in 3.1.3 this is a principal bundle with structure group $L^+_\uparrow \times$ U(1) and will be denoted

simply by $(L_\uparrow^+ \times \mathrm{U}(1))(M)$. Since this fibre-product bundle is built canonically from both bundles $L_\uparrow^+(M)$ and $\mathrm{U}(1)(M)$, it is now possible to decompose ω uniquely into a Lorentzian connection 1-form on $L_\uparrow^+(M)$ and a U(1) potential on $\mathrm{U}(1)(M)$ and represent ω as the sum of these two connection 1-forms (see **Proposition 2** in 3.1.7 for the proof of this general feature of fibre-product bundles). Since a Lorentzian connection 1-form on $L_\uparrow^+(M)$ defines a connection compatible with the metric (cf. 3.2.3), this decomposition of ω would provide the desired separation prescription of the connection (1.11) obtained by McKellar [McK 79].

To make this idea of the pull-back more concrete and to employ it to our complex resultant connection (2.27), let us look at the extended spinor derivative (2.14) and explain its geometric foundation. Before doing so we first consider the usual spinor derivative (2.12): Any metric connection 1-form, to be more precise, any connection 1-form, which defines a Lorentzian connection compatible with the metric, ω_m (with or without torsion) is defined on the Lorentz bundle $L_\uparrow^+(M)$, which — provided that the spacetime M admits a spin structure, cf. 3.2 — is endowed with a "spin structure" $\mathrm{Spin}(M) \twoheadrightarrow L_\uparrow^+(M)$. This is a twofold covering bundle map and induces a $\mathbb{C}^4$-spinor bundle, on which spinors with their spin 1/2 representation are defined properly. With this spin structure, ω_m can be pulled back to $\mathrm{Spin}(M)$ and yields a spin connection, which in turn defines the spinor derivative (2.12) (for details see 3.2.3). On the other hand, a complex linear connection ω, as in our theory, is defined on the whole complex frame bundle $F_c(M)$ built from all tangent frames of $\mathbb{C}\otimes TM$. Since there is no comparable twofold mapping onto $F_c(M)$, ω does not yield a spin connection directly. Therefore, it must be pulled back to an "intermediate bundle", for which an appropriate spin structure exists. Such a bundle is given by $(\mathbb{C}L^+ \times \mathrm{U}(1))(M)$, which is the complex analogue of $(L_\uparrow^+ \times \mathrm{U}(1))(M)$ mentioned above and is built from the complexified orthonormal frame bundle $\mathbb{C}L^+(M)$ and a trivial U(1) bundle $M \times \mathrm{U}(1)$. The fact that ω can indeed be pulled back to this fibre-product, which in itself is not a natural subbundle of the frame bundle, is not as trivial as it might look at first sight, see 3.3.3 for details. Once ω is pulled back onto this intermediate bundle, a complexified spin structure $\mathbb{C}\mathrm{Spin}(M) \twoheadrightarrow \mathbb{C}L^+(M)$ can be employed to pull it back further to $(\mathbb{C}\mathrm{Spin} \times \mathrm{U}(1))(M)$, which then gives rise to the extended spinor derivative (2.14).

According to this geometric background, the resulting connection $\Gamma^a{}_{\mu b}$ (2.27) can be decomposed uniquely into its Lorentzian connection compatible with the metric $\widehat{\Gamma}^a{}_{\mu b}$ on the complex Lorentz bundle $\mathbb{C}L^+(M)$ and a true U(1) potential $S_\mu (= \frac{1}{4}\Gamma^a{}_{\mu a})$ on $M \times \mathrm{U}(1)$, see (3.114), (3.115) and (3.130). It is now clear that the identification $S_\mu = \frac{ie}{\hbar c} A_\mu$ in (2.47) is not only a formal one, but is a true geometric identity on the trivial U(1) bundle $M \times \mathrm{U}(1)$. We can therefore interpret electromagnetism geometrically by choosing S_μ to be the true potential rather than A_μ itself, and describing the electromagnetic interaction through the field equations in (2.46) together with the definite value of l in (2.49) only, thereby completely disregarding

(2.47). This geometrical point of view respects the way in which the U(1) potential S_μ together with the "gravitational" Lorentzian connection (2.28) originated from a single spacetime connection.

Let us stress here that the above discussion of the underlying fibre bundle structure is only a sketch of the more detailed and careful treatment expounded in chapter 3. It should be also noted that this rigorous mathematical treatment is needed for the completion of our theory.

Contrary to other unified field theories mentioned in the introducing chapter, where the whole connection (1.11) is supposed to unify gravity, represented by the Christoffel symbol, and electromagnetism, represented by the torsion trace T_μ, in our theory we see that the non-metric part S_μ must be *detached* from the whole connection on the frame bundle and must be pulled back to the trivial U(1) bundle in order to obtain the electromagnetic potential. This decomposition principle is in accordance with the well-known theorem that it is impossible to combine spacetime and internal symmetry in any but a trivial way [Ora 65]. We can say, however, that it is not necessary to include the electromagnetic potential into the spacetime as something foreign or, as has been done by Infeld and van der Waerden [Inf 33], only on the spin connection level, but that electromagnetic phenomena can be viewed as phenomena originating from the intrinsic geometry of spacetime.

Note that the length scale l (2.49) determining the electromagnetic field strength is very close to the Planck length l_0, which is the characteristic length of quantum gravity. This supports the point of view that gravity and electromagnetism have the same geometrical origin.

2.3.3 Torsion and electromagnetism

We now want to make some clarifying remarks on the identification $T_\mu \sim A_\mu$ (1.10), which has been proposed by many authors so far [Bor 76a, Mof 77, Kun 79, McK 79, Fer 82, Jak 85, Ham 89]. None of them has considered the geometry behind this formal identification.[10]

According to the geometric background briefly outlined in the previous subsection, the true geometric interpretation of electromagnetism is given by

$$S_\mu := \frac{ie}{\hbar c} A_\mu \; . \tag{2.47}$$

Strictly speaking, $S_\mu = \frac{1}{4}\Gamma^a{}_{\mu a} = \frac{ie}{\hbar c}A_\mu$ is not a U(1) potential on the principal bundle $M \times \mathrm{U}(1)$, but a 1-form defined on the spacetime manifold M itself, which has been obtained by pulling back the U(1) potential ω_c on $M \times \mathrm{U}(1)$ onto M via a special

[10] One exception is [Fer 82], in which however a U(1) gauge theory differing from the usual setting was derived. A charged particle is represented by a complexified density on $\bigwedge^4 TM$.

U(1) cross section, namely the trivial cross section $\hat{1}$ defined in (3.98) on p. 53, which prescribes to each point on M the constant value $1 \in \mathrm{U}(1)$:

$$\hat{1}\omega_c = S_\mu dx^\mu = \frac{ie}{\hbar c} A_\mu dx^\mu \ ,$$

see (3.115), where we have omitted the superfluous matrix indices $\delta^a{}_b$. If, instead, another U(1) cross section is used for the pull-back, then it will result in an U(1) gauge transformation of (2.47). To be more presice, if the cross section reads $\exp(\lambda)\hat{1}$, which is a U(1)-valued function assigning to $p \in M$ the value $\lambda(p) \in \mathrm{U}(1)$, this cross section will result in the following gauge transformation (see (3.122))

$$e_a{}^\mu \mapsto e_a{}^\mu \ , \quad S_\mu \mapsto S_\mu + \partial_\mu \lambda \ , \quad \psi \mapsto \exp(\lambda)\psi \ . \tag{2.52}$$

Since this transformation takes place only on the U(1) bundle and on the associated spinor bundle, the tetrad fields and the Lorentzian connection (2.28) as cross sections and as connection 1-form on $\mathbb{C}L^+(M)$ remain invariant, cf. 3.4.

Now, the identification (2.47) can be inserted into the expression of the whole connection (2.27), and its torsion trace can be computed,

$$\begin{aligned} T_\mu &= \Gamma^\alpha{}_{\mu\alpha} - \Gamma^\alpha{}_{\alpha\mu} \ = \ \Gamma^a{}_{\mu a} - \Gamma^a{}_{\alpha b} e_a{}^\alpha e^b{}_\mu \\ &= [\widehat{\Gamma}^a{}_{\mu a} + \delta^a{}_a S_\mu] - [\widehat{\Gamma}^a{}_{\alpha b} + \delta^a{}_a S_\alpha] e_a{}^\alpha e^b{}_\mu \ = \ [4S_\mu] - [-\frac{3}{4} i l_0^2 j_\mu + S_\mu] \\ &= 3\frac{ie}{\hbar c} A_\mu + \frac{3}{4} i l_0^2\, j_\mu \ . \end{aligned} \tag{2.53}$$

This shows that the simple ansatz $T_\mu \sim A_\mu$ is no more valid in our theory, if matter is present. However, since the above equation (2.53) contains both the torsion trace T_μ and the potential A_μ, they still seem to be related to each other. But in contrast to (2.47), the torsion components in (2.53) are derived from the coordinate connection components (2.9), which are obtained by pulling back the general linear connection ω from the frame bundle to M via the cross section given by a coordinate reference frame $(\partial/\partial x^\mu)$. Thus, there is no possibility of an U(1) gauge transformation in (2.53), see (3.135).

If we employ the cross section $\exp(\lambda)\hat{1}$ instead of $\hat{1}$, so that the gauge transformation (2.52) takes place, then the formal application of the formula (2.53) to $S_\mu + \partial_\mu \lambda$ instead of S_μ would result in

$$T_\mu = 3(\frac{ie}{\hbar c} A_\mu + \partial_\mu \lambda) + \frac{3}{4} i l_0^2\, j_\mu \ , \tag{2.54}$$

which is not equal to (2.53). This implies that, to obtain (2.53) from (2.47), the special U(1) gauge $\hat{1}$ implicitly chosen in (2.47) must be held fixed. Since (2.53) is valid in this gauge only, the relation $T_\mu \sim A_\mu$ is merely a formal remnant of the true U(1) identity (2.47).

It is important to note that, in accordance with the decomposition principle explained in 2.3.2, the parallel displacements of (uncharged) vectors and tensors on the spacetime are to be performed with the resultant Lorentzian connection $\widehat{\Gamma}^a{}_{\mu b}$ (2.28) only and not with the full connection $\Gamma^a{}_{\mu b}$. Now, it is easy to see that the torsion trace $\widehat{T}_\mu$ of this Lorentzian connection $\widehat{\Gamma}^a{}_{\mu b}$ does not contain the electromagnetic vector potential, but only the vector current of the Dirac field,

$$\widehat{T}_\mu = \frac{3}{4} i l_0^2 \, j_\mu \, . \tag{2.55}$$

Thus, once the resultant connection $\Gamma^a{}_{\mu b}$ (2.27) is decomposed in its Lorentzian connection part and the U(1) potential part, there is not even a formal relation between the torsion trace and the electromagnetic potential. So, torsion and electromagnetism seem to be two completely different physical quantities, at least in the end. But in order to motivate our theory, especially the extended spinor derivative (2.14), the earlier unified field theories based on the simple ansatz $T_\mu \sim A_\mu$ (1.10) must be considered seriously. We may say that this ansatz is to be viewed as a first formal hint that electromagnetic phenomena originate from the intrisic spacetime geometry.

2.4 Extension of the theory

So far we have considered only an electron. In order to include other, differently charged particles we observe that in the case of a Lorentzian connection with $\Gamma_{a\mu b} = -\Gamma_{b\mu a}$ the spinor derivative (2.12) can be written in many ways

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{4}\Gamma_{a\mu b}\gamma^b\gamma^a\psi = \left(\partial_\mu - \frac{1}{4}\Gamma_{a\mu b}\sigma^{ba} - \frac{1}{4}\Gamma^a{}_{\mu a}\right)\psi \, ; \tag{2.56a}$$

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{1}{4}\Gamma_{a\mu b}\gamma^a\gamma^b\psi = \left(\partial_\mu - \frac{1}{4}\Gamma_{a\mu b}\sigma^{ba} + \frac{1}{4}\Gamma^a{}_{\mu a}\right)\psi \, ; \tag{2.56b}$$

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{4}\Gamma_{a\mu b}\sigma^{ba}\psi \, ; \tag{2.56c}$$

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{4}\Gamma_{a\mu b}\sigma^{ba}\psi + \frac{\varepsilon}{4}\Gamma^a{}_{\mu a}\psi \, , \quad \varepsilon \in \mathbb{R} \, . \tag{2.56d}$$

All these four expressions are equivalent to each other due to the metricity condition.

But if we now insert our complex connection, these four spinor derivatives become different and correspond to derivatives of Dirac spinors with charges $-e$, $+e$, 0, and, more generally, εe, where $\varepsilon \in \mathbb{R}$, respectively. The last case is necessary if fractionally charged particles are considered. Otherwise, the first three cases suffice. Under the U(1) gauge transformation (2.52) the Dirac spinors belonging to each of the above spinor derivatives (2.56a) to (2.56d) transform as

$$S_\mu \mapsto S_\mu + \partial_\mu \lambda \qquad \psi \mapsto \exp(+\lambda)\psi \tag{2.57a}$$

$$S_\mu \mapsto S_\mu + \partial_\mu \lambda \qquad \psi \mapsto \exp(-\lambda)\psi \tag{2.57b}$$
$$S_\mu \mapsto S_\mu + \partial_\mu \lambda \qquad \psi \mapsto \psi \tag{2.57c}$$
$$S_\mu \mapsto S_\mu + \partial_\mu \lambda \qquad \psi \mapsto \exp(-\varepsilon\lambda)\psi \tag{2.57d}$$

see (3.124) and (3.125). To confirm the extension principle in (2.56), according to which we can incorporate Dirac spinors with arbitrary charges into our theory, we will consider a many-particle system in chapter 4.

Chapter 3

Fibre Bundle Geometry

In this chapter we shall fully clarify the underlying fibre bundle structure of the unified field theory discussed in the foregoing chapter. Although the general setting of the fibre bundle background has been already explained in the appendix of the diploma thesis [Hor 94], there are still several important aspects of the fibre geometry which deserve a more detailed consideration.

The salient feature of the bundle structure explained below is that it provides an unique prescription how to construct a spinor derivative out of a given general complex linear connection. The whole construction of the fibre bundle geometry was in fact motivated by the problem of giving the formally extended spinor derivative (2.14) a rigorous mathematical foundation. Without the consideration of Dirac spinors, there would be no clear guideline for the construction of a fibre geometrical background. For example, it is easy to see that in the field equations (2.24) to (2.29) all complex contributions will vanish if no Dirac spinor is present, and the connection will simply be given by the real solution of McKellar (1.11), cf. (2.31a). Also, in this case, the inhomogeneous Maxwell equation (2.29), which follows from (2.23c), would not contain the imaginary unit i, see (2.31b). Thus, as has been said in the discussion following (2.31), a complex geometrical structure would become unnecessary, and this would lead to the conclusion of the important non-metricity vector S_μ being real valued, making the identification (2.47) incorrect. In this case, the question would arise how to make a real valued vector a true U(1) potential.[1]

Note that the consideration of Dirac fields enforces a geometrical decomposition

[1]A solution to this problem has been suggested by Jakubiec and Kijowski in [Jak 85]. Instead of the real valued torsion trace in (1.11) these authors considered the trace ${\Gamma^\alpha}_{\mu\alpha}$ of the whole linear connection as the electromagnetic potential. Since ${\Gamma^\alpha}_{\mu\alpha}$ is not a vector but a connection on the bundle of scalar densities $\bigwedge^4 TM$, it was necessary to introduce the notion of scalar densities of "complex weight" in order to make contact with the commonly used U(1) gauge theory of electromagnetism, see for details [Jak 85]. In other words, they had to introduce the complex structure from the outside. This artificial feature of the geometry is one of the main drawbacks of their theory.

of the whole linear connection (2.27), since otherwise the extended spinor derivative (2.14) remains only a formal definition, and, even worse, could be in contradiction to the well-known twofold representation structure of Dirac fields: A general linear connection, as used in (2.14), can not be "lifted" to a spin connection in contrast to a Lorentzian connection defined on an orthonormal (or Lorentz) frame bundle. The algebraic reason for this fact is that, contrary to the Lorentz group, the structure group $\mathrm{GL}(4, \mathbb{C})$ of the complex frame bundle $F_c(M)$ does not possess a twofold covering map, which accounts for the spin 1/2 nature of spinors. Therefore it is not sufficient to decompose the linear connection only formally, but the whole fibre bundle geometry of such a decomposition must be clarified.

In the first section we discuss some fundamental aspects of the general fibre bundle geometry. This does not mean a mere recapitulation of well-known facts, which can be found in textbooks like [Kob 63], [Gre 72], or [Nak 90], but the discussion comprises several special topics in great detail, which are essential for the construction of the special fibre geometry of our theory, see also [Bos 93, Bos 94]. The topics are the following: principal bundles, bundle mappings, product bundles, associated vector bundles, local cross sections, gauge transformations, and connections and their covariant derivatives.

In the second section the spin geometry is expounded in some detail. The material was gathered from various textbooks [Bau 81, Ber 91, Ble 81, Ben 87, Har 90, Law 89] (see also [Dün 89]) but it also contains own computations. Especially, the notion of a complex spin geometry can be found only en passant in a few textbooks [Ber 91, Har 90, Str 64]. Although most of the structures of the complex spin geometry is merely an exact complexified copy of the real spin geometry, some care is needed because of various possible representations of the complex spin group and of the imbedding of the real structure into the complex one. The complex spin geometry is needed as a central device in the next section, where we develop the fibre bundle geometry of our theory.

The third section contains every detail of the fibre bundle structure needed to complete our geometrical theory of gravity and electromagnetism. The strategy of the construction is as follows: To obtain the "intermediate bundles" (cf. 2.3.2) between the complex frame bundle and a yet unknown spin bundle we first concentrate only on the corresponding structure groups of the principal fibre bundles to be determined. We construct a special diagram of Lie group homomorphisms, which then can easily be translated to a corresponding diagram of bundle mappings. We then use this bundle diagram to map a general complex linear connection 1-form on the frame bundle $F_c(M)$ onto an unique spin connection. To see how this spin connection gives rise to the extended spinor derivative (2.14) we employ the concept of local cross sections to obtain local expressions of the various connections and their covariant derivatives. In doing so we will notice that the resultant non-Lorentzian connection in equation (2.27) can be decomposed unambiguously. Keeping in mind

the main goal of the construction of the special fibre bundle background, namely the unique prescription of building a covariant spinor derivative out of a general complex linear connection, will help to survey these technicalities.

Finally, in the forth section, the U(1) gauge transformation in this geometric setting is explained and compared with the naive gauging of the torsion vector.

3.1 Some aspects of differential geometry

3.1.1 Principal bundles

Let M be a differentiable manifold, which in this work denotes the real 4-dimensional spacetime manifold, although, of course, the following considerations are valid for an arbitrary manifold. Let G be a Lie group. A *principal fibre bundle* over M with group G consists of a manifold $G(M)$ with the following conditions [Kob 63]:

1. The right action, denoted by $G(M) \times G \to G(M)$, $(u, \Lambda) \mapsto u\Lambda$, is free. That is, if $u\Lambda = u$ for some $u \in G(M)$, then Λ is already the trivial element $\Lambda = \mathbb{1} \in G$.

2. Let $\sim$ be the equivalence relation on $G(M)$ defined by $u \sim v \Leftrightarrow u = v\Lambda$ for some (and hence exactly one) $\Lambda \in G$. Then the quotient space $G(M)/_\sim$ is precisely M. If π_G denotes the differentiable canonical projection
$$\pi_G : G(M) \to G(M)/_\sim = M \ , \tag{3.1}$$
then each equivalence class corresponds to exactly one *fibre* $\pi_G^{-1}(p)$, $p \in M$, which is diffeomorphic to G itself, $\pi_G^{-1}(p) \cong G$.

3. Furthermore, every point p has an open neighbourhood $\mathcal{U}$ such that $\pi_G^{-1}(\mathcal{U})$ is *isomorphic* with $\mathcal{U} \times G$ (local triviality). This means that there exists a diffeomorphism
$$\begin{aligned} \Psi = \pi_G \times \phi : \pi_G^{-1}(\mathcal{U}) &\longrightarrow \quad \mathcal{U} \times G \\ u \quad &\longmapsto (\pi\,(u)\,, \phi\,(u))\ , \end{aligned} \tag{3.2}$$
such that $\phi(u\Lambda) = \phi(u) \cdot \Lambda$, where $\Lambda \in G$ and the dot on the left-hand side denotes the group multiplication in G.

We call $G(M)$ the *total space*, M the *base space*, G the *structure group*, and π_G the *(bundle) projection*. If no confusion is to be expected, we will denote the principal bundle simply by $G(M)$. Exceptions are, for example, the bundle of linear frames $F(M)$ and the complex frame bundle $F_c(M)$, whose structure groups are $\mathrm{GL}(4, \mathbb{R})$ and $\mathrm{GL}(4, \mathbb{C})$, respectively.

3.1.2 Bundle mappings

Bundle mappings will be used in 3.3 to pull back linear connection 1-form from the frame bundle $F_c(M)$ to intermediate bundles (see 2.3.2) and, finally, to an extended spin principal bundle. In this way we obtain a special spin connection 1-form, which defines the extended spinor derivative (2.14).

Let $G(M)$ and $H(M)$ denote two principal bundles over the same base manifold M. A *bundle homomorphism* is a pair (f, f_o), where f is a mapping between the total spaces $f : G(M) \to H(M)$ and f_o is a Lie group homomorphism $f_o : G \to H$, where f and f_o must satisfy $f(u\Lambda) = f(u)f_o(\Lambda)$ for all $u \in G(M)$ and $\Lambda \in G$. Here the product $f(u)f_o(\Lambda)$ denotes the right action of H on $H(M)$. This implies that each fibre $\pi_G^{-1}(p)$ of $G(M)$ is mapped into a fibre of $H(M)$. Therefore, a bundle homomorphism (f, f_o) defines a mapping f_M on the base manifold M by $f_M : M \to M$, $p \mapsto \pi_H(f(u))$, where u is an arbitrary element of the fibre $\pi_G^{-1}(p)$.

In this work, we will consider such bundle homomorphisms (f, f_o), which induce the identity mapping $f_M \equiv 1$ on M. Often we denote (f, f_o) simply by f and call it the *bundle mapping*. This means that the following diagram is commutative:

$$\begin{tikzcd}
G(M)\times G \arrow[rr, "f\times f_o"] \arrow[d, "R"] & & H(M)\times H \arrow[d, "R"] \\
G(M) \arrow[rr, "f"] \arrow[dr, "\pi_G"'] & & H(M) \arrow[dl, "\pi_H"] \\
& M &
\end{tikzcd} \tag{3.3}$$

Here R denotes the right group actions.

If, in particular, $f : G(M) \to H(M)$ is an imbedding and $f_o : G \to H$ a Lie group monomorphism, then f is called a *bundle imbedding*. Since f is a topological imbedding, we can identify $G(M)$ with its image $f(G(M))$ and transfer the principal bundle structure of $G(M)$ to $f(G(M))$. This makes $f(G(M))$ itself a principal bundle, which is contained in $H(M)$. We call $f(G(M))$ or $G(M)$ a *(reduced) subbundle* of $H(M)$ and f a *bundle reduction*.

For example, the special Lorentz bundle $L_\uparrow^+(M)$ (p. 8) is a subbundle of the frame bundle $F(M)$, where f is simply the canonical inclusion of $L_\uparrow^+(M)$ in $F(M)$. Also, the frame bundle $F(M)$ is a natural subbundle contained in the complex frame bundle $F_c(M)$.

3.1.3 Product bundles

The notion of product bundles is needed for the construction of the "intermediate bundle", whose very product structure will lead to a natural decomposition of the linear connection (2.27) into its Lorentzian part and its U(1) part in section 3.3.

Let again $G(M)$ and $H(M)$ be two principal bundles over M. Then their (topological) product $G(M) \times H(M)$ is naturally a principal bundle over the base manifold $M \times M$ with structure group $G \times H$. The fibre over a base point $(p, q) \in M \times M$ is given by $\pi_G^{-1}(p) \times \pi_H^{-1}(q)$. Now, if we consider not the whole base space $M \times M$, but only the diagonal space $\Delta := \{(p,p) \in M \times M\}$, then the totality of its fibres, $(\pi_G \times \pi_H)^{-1}(\Delta)$, is easily seen to be a principal bundle again.[2] We identify the diagonal Δ with M itself and denote this $G \times H$ principal bundle by $(G \times H)(M)$,

$$(\pi_G \times \pi_H)^{-1}(\Delta) =: (G \times H)(M) \; . \tag{3.4}$$

Note that in $(G \times H)(M)$ only the fibres are "multiplied". We call this bundle the *(fibre) product bundle*.

An element of $(G \times H)(M)$ is given by $(u, v) \in G(M) \times H(M)$ with the diagonality condition $\pi_G(u) = \pi_H(v)$. If we denote the canonical projections of the total spaces by

$$p: \quad (G \times H)(M) \to G(M) \qquad \text{and} \tag{3.5a}$$

$$q: \quad (G \times H)(M) \to H(M) \quad , \tag{3.5b}$$

and the corresponding canonical projections of the Lie groups $G \times H \to G$ and $G \times H \to H$ by p_o and q_o, respectively, then, (p, p_o) and (q, q_o) define canonical bundle mappings of the fibre product bundle $(G \times H)(M)$ onto its building blocks. But it is important to note that these building blocks $G(M)$ and $H(M)$ are *not* canonical subbundles of $(G \times H)(M)$ in general, that is, it is not always possible to imbed $G(M)$ or $H(M)$ naturally into $(G \times H)(M)$.

3.1.4 Associated vector bundles

Let $G(M)$ be a principal bundle and V a vector space (real or complex), upon which the structure group G acts on the left by a representation ρ:

$$\begin{aligned} G \times V &\longrightarrow \quad V \\ (\Lambda,\, v) &\longmapsto \rho(\Lambda) v \,, \end{aligned} \tag{3.6}$$

[2]This construction is of course valid in a more general setting: If $K(M)$ is a principal bundle over M and $N \subset M$ is a submanifold of M, then $\pi_K^{-1}(N)$ is a principal bundle over N with the same structure group K.

where $\rho(\Lambda)$ is an element of the general linear group of V. Consider now $G(M) \times V$ and introduce an equivalence relation through

$$G(M) \times V \ni (u, v) \sim \Big(u\Lambda,\, \rho(\Lambda^{-1})v\Big)\ , \quad \Lambda \in G\ , \tag{3.7}$$

and denote the resulting quotient space by $V(M)$,

$$V(M) := G(M) \times V/_{\sim} = G(M) \times_\rho V\ . \tag{3.8}$$

An element ϕ of $V(M)$ is thus an equivalence class, which will be denoted by

$$\phi = [u,\, v] = [u\Lambda,\, \rho(\Lambda^{-1})v] \in V(M)\ . \tag{3.9}$$

The manifold $V(M)$ has a canonical projection mapping π_V defined through the principal bundle projection π_G,

$$\begin{aligned} \pi_V : V(M) &\longrightarrow M \\ [u,\, v] &\longmapsto \pi_G(u)\ , \end{aligned} \tag{3.10}$$

where the definition is independent of the choice of representative u. Each point $p \in M$ has an open neighbourhood $\mathcal{U}$ such that $\pi_V^{-1}(\mathcal{U})$ is diffeomorphic to $\mathcal{U} \times V$. The diffeomorphism Ψ_V can be constructed using the local trivialization of the principal bundle $G(M)$. With the help of the isomorphism $\Psi = \pi_G \times \phi$ defined in 3.1.1 we define

$$\begin{aligned} \Psi_V : \pi_V^{-1}(\mathcal{U}) &\overset{\cong}{\longrightarrow} \mathcal{U} \times V \\ [u,\, v] &\longmapsto (\pi_G\,(u)\,,\, \rho\,(\phi\,(u))\, v)\ , \end{aligned} \tag{3.11}$$

which is easily seen to be independent of the choice of representative for the equivalence class.[3] Thus $V(M)$ is a fibre bundle with fibre V, which is called the *vector bundle associated with the principal bundle* $G(M)$.

3.1.5 Local cross sections

Cross sections provide a link between the abstract concept of connection 1-forms defined on a principal bundle and the more familiar notion of connection components on the spacetime manifold. These components lead to a convenient representation of the covariant derivative.

A local cross section σ in a principal bundle $G(M)$ is a mapping from an open subset $\mathcal{U} \subset M$ of the base manifold M to $G(M)$, which respects the fibre structure. That is, for each $p \in \mathcal{U}$ the image $\sigma(p)$ lies in the fibre above p, $\pi_G\,(\sigma\,(p)) = p$.

[3] If we start with another representative of the same equivalence class $[u\Lambda,\, \rho(\Lambda^{-1})v]$ instead, we obtain the same result $\big(\pi_G\,(u\Lambda)\,,\, \rho\,(\phi\,(u\Lambda))\, \rho\big(\Lambda^{-1}\big)\, v\big) = (\pi_G\,(u)\,,\, \rho\,(\phi\,(u))\, v)$.

Given such a local cross section σ, it is possible to trivialize $G(M)$ on $\mathcal{U}$ by defining the following diffeomorphism

$$\begin{aligned} \Psi_\sigma : \pi_G^{-1}(\mathcal{U}) &\stackrel{\cong}{\longrightarrow} \mathcal{U} \times G \\ u &\longmapsto [\pi_G(u),\, \phi_\sigma(u)] \,, \end{aligned} \tag{3.12}$$

where $\phi_\sigma(u) \in G$ is uniquely determined by the definition

$$u =: \sigma\left(\pi_G\left(u\right)\right) \phi_\sigma\left(u\right) \ . \tag{3.13}$$

Since principal bundles are in general not globally trivial, cross sections are normally defined only locally and can not be extended to a global cross section on M.

A cross section v in an associated vector bundle $V(M)$ is defined analogously by demanding $v(p) \in \pi_V^{-1}(p)$. Contrary to the case of principal bundles, any vector bundle admits global cross sections,[4] and these are called *vector fields*. With the help of a local cross section σ over $\mathcal{U}$ in the corresponding principal bundle $G(M)$, a vector field in $V(M)$ can be represented locally by a V-valued function v_σ on $\mathcal{U}$ as follows (see (3.9))

$$\begin{aligned} v|_{\mathcal{U}} : \mathcal{U} &\longrightarrow V(M)|_{\mathcal{U}} \\ p &\longmapsto v(p) = [\sigma(p),\, v_\sigma(p)] \ . \end{aligned} \tag{3.14}$$

As an example, let $V(M)$ be the tangent bundle TM associated to the frame bundle $F(M)$. For the cross section σ of $F(M)$, we take a local coordinate frame (∂_μ). Now, if v is a tangent vector field on TM, then it can be represented by a $\mathbb{R}^4$-valued function v^μ, $\mu = 0, 1, 2, 3$, so that

$$v(p) = [(\partial_\mu|_p),\, v^\mu(p)] \ . \tag{3.15}$$

This is of course nothing but a sophisticated way to express v in its coordinate components via $v = v^\mu \partial_\mu$. The representation (3.14) will be used for the definition of the covariant derivative and also for the local description of a Dirac spinor field ψ below.

3.1.6 Gauge transformation

A gauge transformation (see e.g. [Ble 81]) on a principal bundle $G(M)$ is a special bundle mapping $f : G(M) \to G(M)$, which is a diffeomorphism and induces the identity mapping $f_M = 1$ on M, see 3.1.2. Since f is a diffeomorphism, its corresponding Lie group homomorphism f_o is in fact an isomorphism.

[4] For example, a special cross section is given by the zero cross section, which prescribes to each $p \in M$ the zero vector in its fibre $\pi_V^{-1}(p)$.

Consider now a local cross section σ over $\mathcal{U} \subset M$. For each $p \in \mathcal{U}$, the gauge transformation f acts on σ via $f : \sigma(p) \mapsto f(\sigma(p))$. Since both elements $\sigma(p)$ and $f(\sigma(p))$ lie in the same fibre $\pi_G^{-1}(p)$, there exists an unique local G-valued function $\Lambda : \mathcal{U} \to G$, such that

$$f(\sigma(p)) = \sigma(p)\Lambda(p), \quad p \in \mathcal{U} \ . \tag{3.16}$$

Note that $f(\sigma(p))$ defines another local cross section on $\mathcal{U}$, which we denote by $\tau(p)$. In the foregoing subsection we have represented a vector field v on $V(M)$ by a V-valued function v_σ on $\mathcal{U}$, using the cross section σ on $G(M)$. Analogously, we may define another V-valued function v_τ corresponding to τ via (3.14). With (3.16) and (3.9), we can relate both functions v_σ and v_τ

$$\begin{aligned} v(p) &= [\sigma(p), v_\sigma(p)] = [\sigma(p)\Lambda(p), \rho\left(\Lambda(p)^{-1}\right) v_\sigma(p)] \\ &= [\tau(p), \rho\left(\Lambda(p)^{-1}\right) v_\sigma(p)] \qquad \Rightarrow \\ v_\tau(p) &= \rho\left(\Lambda(p)^{-1}\right) v_\sigma(p) \ . \end{aligned} \tag{3.17}$$

3.1.7 Connections

There are three ways of defining a connection on a principal bundle $G(M)$: The first way is to define it as a special assignment of a subspace Q_u of tangent space $T_uG(M)$ to each point $u \in G(M)$ [Kob 63]. Another way to define a connection is to determine its so-called connection 1-form on the principal bundle $G(M)$ [Kob 63]. Besides these well-known definitions, there is yet another interesting definition (which in turn leads to two other definitions of a connection) based on the notion of the tangential group equivariance [Bos 94]. In our work, we shall employ the second definition.

In this subsection we denote the right action of G on a principal bundle $G(M)$ by R, $u\Lambda =: R_\Lambda(u)$. Let $\boldsymbol{g}$ be the Lie algebra of G and A be an arbitrary element of $\boldsymbol{g}$. Let $\exp(tA)$ be the exponential mapping of A, which defines a path on G. *The fundamental vector field* A^+ on $G(M)$ is defined as follows: If f is a function on $G(M)$, then the action of A^+ on f is determined by $A^+(f) := d/dt|_{t=0} f \circ R_{\exp(tA)}$. Evaluated at a point $u \in G(M)$ this definition means $(A^+(f))(u) = d/dt_{t=0} f(u \exp(tA))$. Since the right action R acts only in the fibres of $G(M)$, but not between different fibres, A^+ is a vector field tangent to the fibres $\pi_G^{-1}(u)$ of $G(M)$.

A connection 1-form ω on a principal bundle $G(M)$ is a 1-form on $G(M)$ with values in the Lie algebra $\boldsymbol{g}$ of G, which satisfies the following conditions:

1. $\omega(A^+) = A$ for each $A \in \boldsymbol{g}$. (3.18a)

2. $R_\Lambda^* \omega = \Lambda^{-1} \omega \Lambda$ for each $\Lambda \in G$. (3.18b)

Here $R^*_\Lambda\omega$ is the pull-back of ω by the right action R_Λ. Explicitly, $(R^*_\Lambda\omega)_u(X) = \omega_{u\Lambda}(R_{\Lambda *}X) = \Lambda^{-1}\omega_u(X)\Lambda$ for a tangent vector X at the point $u \in G(M)$. $R_{\Lambda *}X$ is the push-forward of the vector X and is defined through its action on a function f as follows: $(R_{\Lambda *}X)_{u\Lambda}(f) := X\left(f\left(u\Lambda\right)\right)$.

For the sake of simplicity, from now on we consider only matrix Lie groups, so that in (3.18b) the adjoint mapping $\Lambda^{-1}\omega\Lambda$ can be read simply as a matrix multiplication.

In the third section the following theorem (see [Kob 63]) plays a crucial role:

Proposition 1. Let H be a Lie subgroup of another Lie group G and let $\boldsymbol{h}$ and $\boldsymbol{g}$ be the corresponding Lie algebras, where $\boldsymbol{h}$ is a Lie subalgebra of $\boldsymbol{g}$. Let $H(M)$ and $G(M)$ be principal bundles with structure groups given by H and G, respectively, and suppose that $H(M)$ is a subbundle of $G(M)$.

If there exists a vector subspace $\boldsymbol{m}$ of $\boldsymbol{g}$, such that $\boldsymbol{g}$ can be written as a direct sum (as a vector space) $\boldsymbol{g} = \boldsymbol{h} \oplus \boldsymbol{m}$, and if $\Lambda\boldsymbol{m}\Lambda^{-1} \subset \boldsymbol{m}$ for all $\Lambda \in H$, then from every connection ω on $G(M)$ we can build a connection ω' on $H(M)$ by restring ω to $H(M)$ and taking its $\boldsymbol{h}$-component.

Proof. (See [Kob 63].) To verify the first condition (3.18a) for the connection let A be an element of $\boldsymbol{h}$. Then A is also an element of $\boldsymbol{g}$ and thus $\omega'(A^+) = \omega(A^+) = A$.

Now let Λ be an element of H. To verify the second condition (3.18b) we study $R_\Lambda\omega'$ at a point $u \in H(M)$. Let X be a tangent vector at $T_uH(M)$ and denote the $\boldsymbol{h}$-component of ω by ϖ. Then we have

$$\begin{aligned} R^*_\Lambda\omega'_u(X) + R^*_\Lambda\varpi_u(X) &= (R^*_\Lambda\omega)_u(X) \\ &= \Lambda^{-1}\omega_u(X)\Lambda \\ &= \Lambda^{-1}(\omega'_u(X) + \varpi_u(X))\Lambda \\ &= \Lambda^{-1}\omega'_u(X)\Lambda + \Lambda^{-1}\varpi_u(X)\Lambda\ . \end{aligned}$$

If we now take the $\boldsymbol{h}$-components only, then we obtain the desired result $R^*_\Lambda\omega' = \Lambda^{-1}\omega'\Lambda$ due to the assumption $\Lambda^{-1}\boldsymbol{m}\Lambda \subset \boldsymbol{m}$ in the last line. □

The following proposition (see [Kob 63]) provides the central device for the desired decomposition of our linear connection (2.27):

Proposition 2. Let $(G \times H)(M)$ be the fibre product bundle built from $G(M)$ and $H(M)$. Let ω be a connection 1-form on $(G \times H)(M)$. Then there are unique connection 1-forms ω_G on $G(M)$ and ω_H on $H(M)$ such that $\omega = p^*\omega_G + q^*\omega_H$.

Here p and q denote the canonical bundle mappings defined in 3.1.3.

Proof. Using the trivial projection mappings $p_o : G\times H \to G$ and $q_o : G\times H \to H$ we can decompose ω algebraically as $\omega = p_o\omega + q_o\omega$, where we have used the same letters to denote the Lie group homomorphisms and their Lie algebra homomorphisms. To

define the connection ω_G on $G(M)$, let u be a point in $G(M)$ and X a tangent vector at u. Now, take any point $v \in H(M)$ over the same base point as u, that is $\pi_G(u) = \pi_H(v)$. Then, $(u,v) \in (G \times H)(M)$ and clearly $X \in T_uG(M) \subset T_uG(M) \oplus T_vH(M) = T_{(u,v)}(G \times H)(M)$. We then define

$$\omega_{Gu}(X) := p_o\omega_{(u,v)}(X) \ .$$

To show that this definition does not depend on the particular choice of v, let $v' = v\Lambda$ be another point in $H(M)$ over the same base point, where $\Lambda \in H$. Since X is a vector in the tangent bundle of $G(M)$, it is not affected by the push-forward of the right action R_Λ of the other group H, which can be written on the whole product bundle as $R_{(\mathbb{1},\Lambda)}$. Thus $X = R_{(\mathbb{1},\Lambda)_*}X$, and therefore

$$\begin{aligned} p_o\omega_{(u,v')}(X) &= p_o\omega_{(u,v\Lambda)}\left(R_{(\mathbb{1},\Lambda)_*}X\right) \ = \ p_o[(\mathbb{1},\Lambda)^{-1} \cdot \omega_{(u,v)}(X) \cdot (\mathbb{1},\Lambda)] \\ &= p_o[\mathbb{1}p_o\omega_{(u,v)}(X)\mathbb{1} + \Lambda^{-1}q_o\omega_{(u,v)}(X)\Lambda] \\ &= p_o^2\omega_{(u,v)}(X) \ = \ p_o\omega_{(u,v)}(X) \ . \end{aligned}$$

Using similar techniques it is easy to verify the two conditions for the connection given above. The connection ω_H is defined similarly. It is obvious, that $\omega = p^*\omega_G + q^*\omega_H$. □

The following proposition, which is slightly more general than Proposition 6.2. on page 81 in [Kob 63], is needed for the definition of the spin connection.

Proposition 3. Let $f : G(M) \to H(M)$ be a bundle homomorphism such that its Lie group homomorphism f_o induces a Lie algebra isomorphism, which we denote by the same letter f_o. For every connection ω on $H(M)$, there is an unique connection ω' on $G(M)$ such that $f^*\omega = f_o\omega'$.

Proof. Simply define $\omega' := f_o^{-1}(f^*\omega)$, where f_o^{-1} is the inverse Lie algebra isomorphism. We first prove the condition (3.18a) for connection 1-forms. Let $\boldsymbol{g}$ denote the Lie algebra of G and let $A \in \boldsymbol{g}$. We evaluate $\omega'(A^+)$ at a point $u \in G(M)$ and obtain:

$$\omega'_u(A^+) = f_o^{-1}(f^*\omega)_u(A^+) = f_o^{-1}\omega_{f(u)}\left(f_*A^+(u)\right) \ . \tag{3.19}$$

Since the bundle homomorphism f is not necessarily a diffeomorphism, it is not possible to push-forward the whole fundamental vector field A^+ from $G(M)$ to $H(M)$, but only pointwise. To see what $f_*A^+(u)$ means, we evaluate this vector at $f(u) \in H(M)$ on a smooth function k on $H(M)$. Remembering the definition of the bundle homomorphism (which in this proof does not necessarily induce the identity mapping $f_M = 1$ on the base space) in 3.1.2 and the definition of the fundamental vector fields we obtain

$$f_*(A^+(u))_{f(u)}(k) \ = \ A^+(u)(k \circ f) = \left.\frac{d}{dt}\right|_{t=0} k(f(u\,\exp(tA)))$$

$$
\begin{aligned}
&= \left.\frac{d}{dt}\right|_{t=0} k(f(u)\,\exp(tf_o(A))\,) = f_o(A)^+{}_{f(u)}(k) \quad \Rightarrow \\
f_*(A^+(u)) &= f_o(A)^+{}_{f(u)}\;.
\end{aligned}
$$

Using (3.18a) for ω, we therefore obtain in (3.19) the desired result

$$\omega'_u(A^+) = f_o^{-1}(f^*\omega)_u(A^+) = f_o^{-1}\omega_{f(u)}(f_o(A)^+{}_{f(u)}) = f_o^{-1}(f_o(A)) = A\;.$$

To verify the second condition (3.18b), let $\Lambda \in G$ and X a tangent vector at $u \in G(M)$. Because of the commutative diagram (3.3) (to be more precise, only the rectangle part of it) we have $f \circ R_\Lambda = R_{f_o(\Lambda)} \circ f$, which is important in the following computation

$$
\begin{aligned}
\Big(R_\Lambda^*(f_o^{-1}f^*\omega)\Big)_u(X) &= f_o^{-1}\omega_{f(u\Lambda)}\left((f\circ R_\Lambda)_*X\right) \\
&= f_o^{-1}\omega_{f(u)f_o(\Lambda)}\Big((R_{f_o(\Lambda)}\circ f)_*X\Big) \\
&= f_o^{-1}(R^*_{f_o(\Lambda)}\omega)_{f(u)}(f_*X) \\
&= f_o^{-1}(f_o(\Lambda^{-1})\omega_{f(u)}(f_*X)f_o(\Lambda)) \\
&= \Lambda^{-1}\cdot f_o^{-1}(f^*\omega)_u(X)\cdot\Lambda\;.
\end{aligned}
$$

This completes the proof. $\square$

Note the important fact that the Lie group homomorphism f_o needs not to be an isomorphism, but only its concommitant Lie algebra mapping. Thus the group homomorphism can be a twofold mapping, which is the case for the universal covering map of the Lorentz group by its spin group SL(2, $\mathbb{C}$), see below at 3.2.2.

3.1.8 Covariant derivatives

Given a connection ω on a principal bundle $G(M)$, we now construct its covariant derivative on the associated vector bundle $V(M)$. For our purposes it is convenient to define it by using a local cross section. A fairly detailed account of this topic can be found for example in [Nak 90]. In the following, proofs are omitted in order to keep the presentation lucid.

Let v be a vector field in $V(M)$. From (3.14) we know that v can be represented by a V-valued function v_σ, when a local cross section σ is given on $\mathcal{U}$ in $G(M)$,

$$v = [\sigma,\, v_\sigma]\;.$$

Let X be a tangent vector at $p \in \mathcal{U} \subset M$. We then define the *covariant derivative* $\nabla_X v$ *of* v *at* p *in the direction* X as follows

$$\nabla_X v := [\sigma(p),\, X(v_\sigma(p)) + \rho\,(\sigma^*\omega(X))\,v_\sigma(p)]\;, \tag{3.20}$$

where $\sigma^*\omega$ is the pull-back of the connection 1-form by σ, and the same symbol ρ is used to denote the Lie algebra homomorphism defined by the representation ρ of the structure group G into V. Thus, $\rho(\sigma^*\omega)$ is a 1-form defined on $\mathcal{U}$ with values in the Lie algebra of the general linear group of V.

It can be shown that this definition of the covariant derivative is equivalent to the other, more common, definition which is directly built on the notion of parallel displacements of vector fields, see e.g. [Nak 90]. Here we are not going to prove this equivalence, since the proof is very technical, but we show that the definition (3.20) is independent of the special choice of the local cross section σ.

Let τ be another local cross section. Since for each $p \in \mathcal{U}$ the values $\sigma(p)$ and $\tau(p)$ lie in the same fibre over p, we can find a G-valued function Λ such that

$$\tau(p) = \sigma(p)\Lambda(p) \,. \tag{3.21}$$

Before pulling back ω by τ, we evaluate the push-forward τ_*X of the tangent vector X at the base point $p \in M$. In (3.21) let τ_0, σ_0, and Λ_0 denote the values of the corresponding fields at this fixed point p. Using the Leibniz rule we can calculate the action of τ_*X on a function f on $G(M)$ as follows

$$\begin{aligned}
(\tau_*X)_{\tau(p)}f &= X_p(f(\tau)) \;=\; X_p(f(\sigma\Lambda)) \\
&= X_pf(\sigma\Lambda_0) + X_pf(\sigma_0\Lambda) \\
&= X_pf(R_{\Lambda_0}\circ\sigma) + X_pf(\tau_0\Lambda_0^{-1}\Lambda)\,.
\end{aligned} \tag{3.22}$$

In the last line the first term can be simply expressed as

$$X_pf(R_{\Lambda_0}\circ\sigma) = (\sigma_*X)_{\,\sigma_0}(f\circ R_{\Lambda_0}) = (R_{\Lambda_0*}(\sigma_*X))_{\,\tau_0}f \,. \tag{3.23}$$

Note that this equation would be incorrect if Λ_0 was not constant. To bring the second term in (3.22) into a more familiar form, let us introduce a curve $\gamma(t)$ in M, which runs through $p = \gamma(0)$ and whose tangent vector $\frac{d}{dt}|_{t=0}\gamma$ is precisely the vector X. Then this curve defines a curve in the structure group G via $\Lambda(\gamma(t))$, whose derivative $\frac{d}{dt}|_{t=0}\Lambda(\gamma(t))$ is equal to $X(\Lambda)$. We obtain

$$\begin{aligned}
X_pf(\tau_0\Lambda_0^{-1}\Lambda) &= \left.\frac{d}{dt}\right|_{t=0} f(\tau_0\Lambda(\gamma(0))^{-1}\Lambda(\gamma(t))\,) \\
&= \left.\frac{d}{dt}\right|_{t=0} f(\tau_0\exp(t\Lambda(\gamma(0))^{-1}X(\Lambda))\,) \qquad (3.24)\\
&= (\Lambda^{-1}X(\Lambda))^+_{\tau_0}f \,, \qquad (3.25)
\end{aligned}$$

where we note that in (3.24) the curve $\Lambda(\gamma(0))^{-1}\Lambda(\gamma(t))$ runs through $\mathbb{1} \in G$ and has, at this point, the same tangent as $\exp(t\Lambda(\gamma(0))^{-1}\frac{d}{dt}|_{t=0}\Lambda(\gamma(t))\,)$, which defines a fundamental vector field as in (3.25). With the help of (3.22) to (3.25) we obtain

$$\tau_*X = R_{\Lambda*}(\sigma_*X) + (\Lambda^{-1}X(\Lambda))^+ \,, \tag{3.26}$$

where we have omited the point of evaluation. With this result we can relate $\tau^*\omega$ and $\sigma^*\omega$ immediately as follows, using the characteristic conditions (3.18) for the connection 1-form ω

$$\begin{aligned} \tau^*\omega(X) &= \omega(\tau_* X) \\ &= \omega(R_{\Lambda *}(\sigma_* X)) + \omega((\Lambda^{-1}X(\Lambda))^+) \\ &= \Lambda^{-1}\omega(\sigma_* X)\Lambda + \Lambda^{-1}X(\Lambda) \\ &= \Lambda^{-1}\sigma^*\omega(X)\Lambda + \Lambda^{-1}X(\Lambda)\ . \end{aligned} \tag{3.27}$$

This formula displays the gauge transformation law of the connection 1-form. To see that the covariant derivative defined in (3.20) is independent of the choice of the cross section, we use (3.17) to obtain first

$$\begin{aligned} X(v_\tau) &= X(\rho(\Lambda^{-1})v_\sigma) \\ &= \rho(X(\Lambda^{-1}))v_\sigma + \rho(\Lambda^{-1})X(v_\sigma) \\ &= \rho(\Lambda^{-1})(\rho(\Lambda X(\Lambda^{-1})) + X(v_\sigma)) \\ &= \rho(\Lambda^{-1})(\,-\,\rho(\Lambda^{-1}X(\Lambda)) + X(v_\sigma))\ . \end{aligned} \tag{3.28}$$

Finally, if we use the cross section τ in the definition (3.20), then from (3.27), (3.28), and (3.17) we obtain the desired invariant result:

$$\begin{aligned} \nabla_X v &= [\tau(p)\ ,\ X(v_\tau(p)) + \rho((\tau^*\omega)(X))v_\tau(p)] \\ &= [\sigma(p)\Lambda(p)\ ,\ \rho(\Lambda(p)^{-1})(X(v_\sigma(p)) + \rho(\sigma^*\omega(X))v_\sigma(p))] \\ &= [\sigma(p)\ ,\ X(v_\sigma(p)) + \rho(\sigma^*\omega(X))v_\sigma(p)]\ . \end{aligned} \tag{3.29}$$

3.2 Spin geometry

In this section we will discuss the spin structure over a spacetime manifold M with a Lorentzian (pseudo-Riemannian) metric $g_{\mu\nu}$. The spin structure consists of a so-called spin bundle over M and a twofold bundle map from this spin bundle to the special Lorentz bundle introduced on p. 8, and is necessary in order to introduce Dirac spinors on a curved manifold. Contrary to this "real spin structure", its complexified version, which we call "complex spin structure", is less well-known but is needed for our theory, since, roughly speaking, the linear connection determining the covariant derivative of the Dirac spinors is complex in our theory. This complexification is, as we will see, rather straightforward if written in local representations using tensor components[5] but is not so trivial if the underlying global geometry is

[5]For example, Ashtekar's formalism of general relativity [Ash 91] employs a complexified theory of general relativity. Although spinor fields are considered in this complex geometry, the notion of a complex spin structure is absent, because, working with local representations, a recourse to the complex spin structure is not necessary, since all computations go through by simply allowing the real variables to be complex valued.

taken into account.

3.2.1 Spin structure of Minkowski spacetime

Let us first consider Minkowski spacetime and its familiar spin structure, which will be generalized to the case of an arbitrary curved spacetime in the next subsection.

The special orthochronous Lorentz group $L_\uparrow^+$ was defined in (2.1) as

$$L_\uparrow^+ = \{\Lambda \in \mathrm{Mat}(4, \mathbb{R}) | \Lambda^T \eta \Lambda = \eta,\ \det \Lambda = 1,\ \Lambda^0{}_0 \geq 1\} . \tag{3.30}$$

The spin group "Spin" of $L_\uparrow^+$ is, by definition, the universal covering space of $L_\uparrow^+$, which is simply given by Spin $= \mathrm{SL}(2, \mathbb{C})$. The covering map is a twofold Lie group homomorphism, denoted by

$$\xi_o : \mathrm{Spin} \cong \mathrm{SL}(2, \mathbb{C}) \longrightarrow L_\uparrow^+ . \tag{3.31}$$

To make this map ξ_o more explicit, let x^a be the cartesian components of a point in flat Minkowski spacetime. We define the following vector space isomorphism from the Minkowski space to the vector space of hermitian 2×2 matrices,

$$\sim : x = (x^a) \longmapsto \widetilde{x} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} . \tag{3.32}$$

Then the action of the Lorentz matrix $\xi_o(A)$, where $A \in \mathrm{SL}(2, \mathbb{C})$, is defined by

$$\widetilde{\xi_o(A)(x)} := A(\widetilde{x})A^\dagger . \tag{3.33}$$

This homomorphism ξ_o is twofold, since, obviously, both A and $-A$ result in the same Lorentz map.

By definition, a covering map is locally a diffeomorphism. Especially, the spin map ξ_o (3.31) induces an isomorphism between the Lie algebras of $\mathrm{SL}(2, \mathbb{C})$ and $L_\uparrow^+$, which we shall now state explicitly.

Let $D_0{}^a$, $a = 1, 2, 3$, denote the infinitesimal generators of the Lorentz boosts in the three coordinate directions x^a. Let $D_a{}^b$, $a, b = 1, 2, 3$ and $a \neq b$, be the infinitesimal generators of ordinary rotations of space, whose rotation axes are given by $\pm x^c$, where c and the rotation direction $\pm$ are determined by demanding that the triplet (a, b, c) should be a positive (negative) permutation of $(1, 2, 3)$; in this way, there are, up to sign, of course only three generators of rotations.[6] Altogether, there are 6 independent generators of the Lie algebra $\boldsymbol{l}$ (2.1) of the Lorentz group.

[6] For example $D_0{}^1 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $D_3{}^2 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$.

With the help of the canonical generators $E_a{}^b$ of the whole matrix group $\mathrm{Mat}(4, \mathbb{R})$, whose entries are 1 in the a-th row of the b-th column and 0 otherwise, that is,

$$(E_a{}^b)^c{}_d = \delta^c{}_a \, \delta^b{}_d \ , \tag{3.34}$$

we can write all Lorentz generators systematically as

$$D_a{}^b = \frac{1}{2} \left(E_a{}^b - \eta_{ac} E_d{}^c \eta^{db} \right) \ . \tag{3.35}$$

Here, a and b are allowed to take any value 0, 1, 2, 3, but only those combinations satisfying $a \neq b$ yield non-zero results.

On the other hand, the Lie algebra of $\mathrm{SL}(2, \mathbb{C})$, denoted by $\boldsymbol{sl}(2, \mathbb{C})$, has the 6 generators (to be more precise, generators of the real algebra) given by the three Pauli matrices and their imaginary multiples,

$$\sigma^1, \ \sigma^2, \ \sigma^3, \ i\sigma^1, \ i\sigma^2, \ i\sigma^3 \ , \tag{3.36}$$

where the Pauli matrices are given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{3.37}$$

The Lie algebra isomorphism induced by ξ_o, which we will denote by the same letter, is determined through the following relations between the generators of the Lie algebras $\boldsymbol{sl}(2, \mathbb{C})$ and $\boldsymbol{l}$:

$$\begin{aligned} &\sigma^a \longmapsto 4D_0{}^a, \quad a = 1, 2, 3 \ , \\ &i\sigma^c \longmapsto 4D_a{}^b, \quad (a, b, c) = (1, 2, 3), \ (2, 3, 1), \ (3, 1, 2) \ . \end{aligned} \tag{3.38}$$

This result can be directly deduced from the definition (3.33) of the spin map ξ_o.

In Minkowski spacetime, Dirac spinors ψ are vector fields with values in $\mathbb{C}^4$, which, however, do not obey the ordinary transformation law of vectors. If, for example, the Minkowski spacetime is rotated by a Lorentz matrix Λ, then an ordinary vector X^a at a spacetime point x^a is transformed into $\Lambda^a{}_b X^b$ at $\Lambda^a{}_b x^b$. On the other hand, a Dirac spinor ψ is transformed according to the following law: Let A be one of the two elements in $\mathrm{SL}(2, \mathbb{C})$, which is mapped by ξ_o onto Λ. Then ψ transforms as

$$\psi \longmapsto \zeta(A)\psi := \begin{pmatrix} A & 0 \\ 0 & (A^\dagger)^{-1} \end{pmatrix} \psi \ , \tag{3.39}$$

where

$$\begin{aligned} \zeta : \mathrm{SL}(2, \mathbb{C}) &\longrightarrow \quad \mathrm{GL}(4, \mathbb{C}) \\ A \quad &\longmapsto \begin{pmatrix} A & 0 \\ 0 & (A^\dagger)^{-1} \end{pmatrix} , \end{aligned} \tag{3.40}$$

is called a *spin representation*. The peculiar feature of the transformation law (3.39) is, that, if the Minkowski spacetime is rotated gradually from 0° up to 360°, so that vectors retain their original values again, a Dirac spinor ψ will be transformed into $-\psi$. This characteristic spin 1/2 behaviour of spinors is due to the twofold spin homomorphism ξ_o.

In the following, we need an explicit expression of the Lie algebra homomorphism $\zeta \circ \xi_o^{-1}$ from the Lie algebra $\boldsymbol{l}$ of the Lorentz group into the Lie algebra $\boldsymbol{gl}(4, \mathbb{C})$ of $\mathrm{GL}(4, \mathbb{C})$. This is most easily displayed by using the generators of the Lorentz algebra defined above, where we only give the result and omit the derivation:

$$\zeta \circ \xi_o^{-1} : \boldsymbol{l} \longrightarrow \boldsymbol{gl}(4, \mathbb{C}) , \quad D_a{}^b \longmapsto -\frac{1}{4}\gamma^b\gamma_a , \tag{3.41}$$

where $a \neq b$ is to be understood. Note that the factor 1/4 arises from the inverting of the factor 4 in (3.38).

3.2.2 Real spin geometry

Let $(M, g_{\mu\nu})$ be the spacetime manifold with a Lorentzian (pseudo-Riemannian) metric. A *spin structure* is a copy of all the structures discussed so far for a flat Minkowski spacetime to the case of a non-flat spacetime M. Now the spin homomorphism ξ_o is replaced by a bundle map ξ and the Dirac spinors become cross sections of a spinor bundle $S(M)$. However, these structures can only be defined if some global topological conditions are met by the manifold M. For the sake of simplicity, we will assume that this is the case.[7]

Let $L_\uparrow^+(M)$ be the special Lorentz bundle introduced on p. 8. It is a $L_\uparrow^+$ principal bundle and it is built from certain orthonormal tangent frames in TM. A spin bundle $\mathrm{Spin}(M)$ is a principal bundle with structure group $\mathrm{Spin} = \mathrm{SL}(2, \mathbb{C})$, which is defined together with the bundle map $\xi : \mathrm{Spin}(M) \rightarrow L_\uparrow^+(M)$ via the following commutative diagram (cf. 3.1.2):

[7] For a detailed discussion of the topological conditions we refer to [Bau 81, Law 89, Ger 68, Ger 70].

$$
\begin{tikzcd}
\mathrm{Spin}(M) \times \mathrm{Spin} \arrow[rr, "\xi \times \xi_o"] \arrow[d, "R"] & & L^+_\uparrow(M) \times L^+_\uparrow \arrow[d, "R"] \\
\mathrm{Spin}(M) \arrow[rr, "\xi"] \arrow[dr, "\pi"'] & & L^+_\uparrow(M) \arrow[dl, "\pi"] \\
& M &
\end{tikzcd}
\tag{3.42}
$$

We call this bundle mapping ξ a *spin structure*.[8] According to (3.42), the bundle map ξ is compatible with the spin map ξ_o (3.31), that is,

$$\xi(uA) = \xi(u)\xi_o(A), \quad u \in \mathrm{Spin}(M), \quad A \in \mathrm{SL}(2, \mathbb{C}) \; . \tag{3.43}$$

Since the spin structure ξ can be replaced by the spin map ξ_o in each fibre of $\mathrm{Spin}(M)$, ξ is also a twofold covering map and therefore surjective.

Once such a spin structure (3.42) is given, we can define the *spinor bundle* $S(M)$, which is the $\mathbb{C}^4$ vector bundle associated to $\mathrm{Spin}(M)$, the representation of the structure group Spin precisely being the spin representation ζ of (3.40). Thus,

$$S(M) = \mathrm{Spin}(M) \times_\zeta \mathbb{C}^4 \; , \tag{3.44}$$

see (3.8). Cross sections into this associated vector bundle $S(M)$ are called *Dirac spinors*, see e.g. [Ble 81].

We shall now define the covariant spinor derivative built from a Lorentzian connection compatible with the metric $g_{\mu\nu}$, cf. (2.11). According to (3.20) a covariant spinor derivative can be constructed from a connection on the spin bundle $\mathrm{Spin}(M)$, which is called a *spin connection*.[9] So the only task is to obtain a spin connection from a Lorentzian connection. But from **Proposition 3** it follows that this is indeed possible: Since the Lie algebra homomorphism ξ_o is actually an isomorphism, see (3.38), the spin structure ξ in (3.42) yields a spin connection ω_s starting from *any* metric connection 1-form ω_m defined on the Lorentz bundle $L^+_\uparrow(M)$,

$$\omega_s = \xi_o^{-1}(\xi^* \omega_m) \; . \tag{3.45}$$

[8] Note that there may be more than one spin structure for a given Lorentz bundle, see [Bau 81].

[9] We remark that the converse statement is not true: A covariant derivative on an associated vector bundle can be defined without the notion of connection 1-forms, and there might exist a covariant derivative, which can not be derived from a connection 1-form on the principal bundle via (3.20), see [Gre 72].

3.2.3 Covariant spinor derivative

In the following we shall study two questions: First, how does a metric connection 1-form ω_m define a Lorentzian connection $\Gamma^a{}_{\mu b}$ on the spacetime manifold M? Secondly, how does this connection yield the covariant spinor derivative (2.12)?

According to our definition (3.20) of the covariant derivative, we first of all need a local cross section σ in the Lorentz bundle $L^+_\uparrow(M)$, that is, an orthonormal tetrad field $(e_a{}^\mu)$. Then, since the spin structure is a surjective mapping, there exists a local cross section $\hat{\sigma}$ in the spin bundle $\mathrm{Spin}(M)$, such that

$$\xi(\hat{\sigma}) = \sigma \ . \tag{3.46}$$

In fact there are exactly two such cross sections in $\mathrm{Spin}(M)$, namely $\hat{\sigma}$ and $\hat{\sigma}(-\mathbb{1})$. The metric connection 1-form ω_m on $L^+_\uparrow$ can be pulled back to the base manifold M via σ yielding a matrix-valued 1-form, whose components are defined as

$$\Gamma^a{}_{\mu b} dx^\mu := (\sigma^* \omega_m)^a{}_b \ . \tag{3.47}$$

Since these components belong to a matrix of the Lie algebra $\boldsymbol{l}$ of the Lorentz group, they satisfy, according to (2.1),

$$\Gamma^c{}_{\mu b}\eta_{ca} + \eta_{bc}\Gamma^c{}_{\mu a} = \Gamma_{a\mu b} + \Gamma_{b\mu a} = 0 \ , \tag{3.48}$$

which is precisely the metricity condition (2.11). Instead of giving the matrix components of the connection like in (3.47) we can of course give the full $\boldsymbol{l}$-valued connection 1-form on M by using the canonical generators $E^a{}_b$, see (3.34), and the Lorentz generators (3.35) as follows

$$\begin{aligned} \sigma^*\omega_m &= \Gamma^a{}_{\mu b}\, dx^\mu \cdot E_a{}^b = \frac{1}{2}(\Gamma^a{}_{\mu b} - \Gamma_{b\mu}{}^a) \cdot E_a{}^b\, dx^\mu \\ &= \Gamma^a{}_{\mu b} \cdot \frac{1}{2}\left(E_a{}^b - \eta_{ac}E_d{}^c\eta^{db}\right)\, dx^\mu \\ &= \Gamma^a{}_{\mu b} \cdot D_a{}^b\, dx^\mu \ . \end{aligned} \tag{3.49}$$

This expression will be used below to derive the covariant spinor derivative.

To define the covariant derivative of a vector field $v = v^\mu \partial_\mu$ in the tangent bundle TM, we represent it by a $\mathbb{R}^4$-valued function v^a, see (3.14),

$$v = [\sigma,\, v^a] = [(e_a{}^\nu \partial_\nu),\, v^a] \ , \tag{3.50}$$

so that v^a are merely the anholonomic tetrad components of the vector field v. With the definition (3.20), the covariant derivative of v now reads

$$\nabla_\mu v = [(e_a{}^\nu \partial_\nu)\ ,\ \partial_\mu v^a + \Gamma^a{}_{\mu b} v^b] \ , \tag{3.51}$$

which is commonly and more loosely written as

$$\nabla_\mu v^a = \partial_\mu v^a + \Gamma^a{}_{\mu b} v^b \ . \tag{3.52}$$

This is exactly the usual covariant derivative of a vector field in orthonormal, anholonomic components, showing together with (3.48) that the metric connection 1-form ω_m indeed defines a Lorentzian connection (in anholonomic components) via (3.47).

We now define the covariant spinor derivative with the help of this metric connection ω_m. In close analogy to the case of the vector field v before, we first represent a Dirac field ψ by a $\mathbb{C}^4$-valued function $\psi_{\hat\sigma}$ via the cross section $\hat\sigma$ (3.46)

$$\psi = [\hat\sigma,\, \psi_{\hat\sigma}] \ . \tag{3.53}$$

We then employ the spin connection (3.45) to define the covariant spinor derivative

$$\nabla_\mu \psi = [\hat\sigma \ , \ \partial_\mu \psi_{\hat\sigma} + \zeta(\hat\sigma^* \omega_s)(\partial_\mu)\psi_{\hat\sigma}] \ , \tag{3.54}$$

where

$$\begin{aligned} \zeta(\hat\sigma^*\omega_s)(\partial_\mu)\psi_{\hat\sigma} &= \zeta((\hat\sigma^*\xi_o^{-1}(\xi^*\omega_m))(\partial_\mu))\psi_{\hat\sigma} \\ &= \zeta \circ \xi_o^{-1}(\xi(\hat\sigma)^*\omega_m(\partial_\mu))\psi_{\hat\sigma} \\ &= \zeta \circ \xi_o^{-1}(\sigma^*\omega_m(\partial_\mu))\psi_{\hat\sigma} \\ &= \zeta \circ \xi_o^{-1}(\Gamma^a{}_{\mu b} D_a{}^b)\psi_{\hat\sigma} \\ &= -\frac{1}{4}\gamma^b\gamma_a\Gamma^a{}_{\mu b}\psi_{\hat\sigma} \ . \end{aligned} \tag{3.55}$$

(This is derived with the help of (3.49) and (3.41).) The result may now be written as

$$\nabla_\mu \psi_{\hat\sigma} = \partial_\mu \psi_{\hat\sigma} - \frac{1}{4}\Gamma^a{}_{\mu b}\gamma^b\gamma_a\,\psi_{\hat\sigma} \ . \tag{3.56}$$

Usually, the subscript $\hat\sigma$, denoting the special cross section used to represent ψ as a $\mathbb{C}^4$-valued function, is skipped. We see that this covariant derivative is precisely the one given in (2.12).

In summary, we have exploited the spin structure (3.42) to obtain a covariant spinor derivative out of an arbitrary metric connection 1-form ω_m. Note that we have never spoken of the Levi–Civita connection. In fact, the metric connection may have non-vanishing torsion.

3.2.4 Complex spin geometry

We now introduce the notion of complex spin geometry. This complex extension is necessary in order to accomodate the real spin structure to the complex tangent bundle geometry used in our theory.

Algebraic preliminaries

It is well-known that the full (real) Lorentz group L,

$$L := \{\Lambda \in \mathrm{GL}(4,\mathbb{R}) \,|\, \Lambda^t \eta \Lambda = \eta\} \;, \tag{3.57}$$

consists of four topological components characterized by the sign of the determinant and the sign of the component $\Lambda^0{}_0$. The complex Lorentz group $\mathbb{C}L$ is defined analogously

$$\mathbb{C}L := \{\Lambda \in \mathrm{GL}(4,\mathbb{C}) \,|\, \Lambda^t \eta \Lambda = \eta\} \;. \tag{3.58}$$

Contrary to L however, $\mathbb{C}L$ consists of only two components, because those components of L, which are separated by the sign of $\Lambda^0{}_0$ are now connected by a path over complex Lorentz matrices. The two components of $\mathbb{C}L$ are characterized by the sign of the determinant only, see for a detailed discussion [Str 64]. The special complex Lorentz group $\mathbb{C}L^+$ is the component containing $\mathbb{1}$,

$$\mathbb{C}L^+ := \{\Lambda \in \mathrm{GL}(4,\mathbb{C}) | \Lambda^t \eta \Lambda = \eta \,,\; \det(\Lambda) = 1\} \;. \tag{3.59}$$

Contrary to the real case, where $L^+_\uparrow$ is of course not isomorphic to SO(4), the special complex Lorentz group $\mathbb{C}L^+$ is isomorphic to the complex special orthogonal group $\mathbb{C}\mathrm{SO}(4)$, which is defined analogously to the real case, $\mathbb{C}\mathrm{SO}(4) = \{\Lambda' \in \mathrm{GL}(4,\mathbb{C}) \,|\, \Lambda'^T \Lambda' = \mathbb{1}\}$. The isomorphism is given by $\Lambda' = W^{-1} \Lambda W$, where $W = \mathrm{diag}(i,1,1,1)$ is simply the Wick-rotation.

Since $\mathbb{C}L^+$ is a complex 6-dimensional Lie group, it has twice as much real dimensions as the real Lorentz group $L^+_\uparrow$. Correspondingly, the spin group of $\mathbb{C}L^+$ has also 12 real dimensions and is given by [Str 64]

$$\mathbb{C}\mathrm{Spin} := \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \;. \tag{3.60}$$

The twofold spin map will be denoted by the same letter ξ_o as in the real case (3.31),

$$\xi_o : \mathbb{C}\mathrm{Spin} \cong \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathbb{C}L^+ \;, \tag{3.61}$$

and is now defined by[10] (compare (3.33))

$$\xi_o((\widetilde{A,B}))(x) := A(\widetilde{x})B^\dagger \;, \tag{3.62}$$

where $(A,B) \in \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. That (3.62) indeed defines a complex Lorentz matrix can be seen as follows: The Lorentz group is characterized by the transformation invariance of the metric measure, which is given by (see (3.32))

$$x^T \eta x = \det(\widetilde{x}) \;. \tag{3.63}$$

[10] Our convention differs from that used in [Str 64].

This yields the desired invariance property[11]

$$(\xi_o((A,B))(x))^T\eta(\xi_o((A,B))(x)) = \det(A(\tilde{x})B^\dagger) = 1\cdot\det(\tilde{x})\cdot 1\ . \tag{3.64}$$

Instead of the 6 Lorentz generators in (3.35) there are now 12 generators of $\mathbb{C}L^+$ given by

$$D_a{}^b \quad \text{and} \quad iD_a{}^b\ . \tag{3.65}$$

There are also 12 generators of the complex spin group $\mathbb{C}$Spin, and these are mapped onto the Lorentz generators by the Lie algebra isomorphism ξ_o as follows:

$$\begin{array}{ll} (\sigma^a,\sigma^a) & \longmapsto 4D_0{}^a,\ \ a=1,2,3\ , \\ (i\sigma^\gamma,i\sigma^c) & \longmapsto 4D_a{}^b,\ \ (a,b,c)=(1,2,3),\ (2,3,1),\ (3,1,2)\ , \\ (\sigma^c,-\sigma^c) & \longmapsto -4iD_a{}^b,\ \ (a,b,c)=(1,2,3),\ (2,3,1),\ (3,1,2)\ , \\ (i\sigma^a,-i\sigma^a) & \longmapsto 4iD_0{}^a,\ \ a=1,2,3\ . \end{array} \tag{3.66}$$

The complex spin representation ζ of $\mathbb{C}$Spin into $\mathrm{GL}(4,\mathbb{C})$ is defined as

$$\begin{array}{rcl} \zeta:\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C}) & \longrightarrow & \mathrm{GL}(4,\mathbb{C}) \\ (A,B) & \longmapsto & \begin{pmatrix} A & 0 \\ 0 & (B^\dagger)^{-1}\end{pmatrix}, \end{array} \tag{3.67}$$

where we have used the same letter ζ as in the real spin representation (3.40). According to this complex representation, Dirac spinors transform as

$$\psi \longmapsto \begin{pmatrix} A & 0 \\ 0 & (B^\dagger)^{-1}\end{pmatrix}\psi\ . \tag{3.68}$$

If we now look at both the complexified Lie group homomorphisms ξ_o and the complex spin representation ζ, then these two maps are constructed in such a pleasant way that the resultant Lie algebra homomorphism $\zeta\xi_o^{-1}$ from the complex Lorentz Lie algebra $\mathbb{C}\otimes\boldsymbol{l}$ into $\boldsymbol{gl}(4,\mathbb{C})$ is the same as the real homomorphism (3.41), that is,

$$\zeta\circ\xi_o^{-1}:\mathbb{C}\otimes\boldsymbol{l}\longrightarrow\boldsymbol{gl}(4,\mathbb{C})\,,\quad D_a{}^b\longmapsto -\frac{1}{4}\gamma^b\gamma_a\ . \tag{3.69}$$

So, especially,

$$iD_a{}^b\longmapsto -i\frac{1}{4}\gamma^b\gamma_a\ . \tag{3.70}$$

[11] To prove that $\xi_o((A,B))$ has positive determinant, as is required by the definition of $\mathbb{C}L^+$, let us notice $\xi_o((\mathbb{1},\mathbb{1}))=\mathbb{1}$. This means that there is one point in $\mathbb{C}$Spin which is mapped into $\mathbb{C}L^+$. Since $\mathbb{C}$Spin is definitely connected (because $\mathrm{SL}(2,\mathbb{C})$ is connected), ξ_o maps the whole domain group $\mathbb{C}$Spin into $\mathbb{C}L^+$, proving the assertion. That ξ_o is actually surjective can be seen using familiar topological arguments.

Bundle analogue

Precisely as in the real spin geometry, the complex spin geometry is an exact translation of the complex spin algebra into the framework of fibre bundles. Instead of the real Lorentz bundle $L^{+}_{\uparrow}(M)$, we now have its complexified version, namely a complex Lorentz bundle $\mathbb{C}L^{+}(M)$, which not only contains real orthonormal tangent frames of TM, but also complex orthonormal tangent frames of $\mathbb{C} \otimes TM$. The structure group is $\mathbb{C}L^{+}$.

The *complex spin structure* will be denoted by the same letter ξ as in the real case and consists of a complex spin bundle $\mathbb{C}\mathrm{Spin}(M)$ with structure group $\mathbb{C}\mathrm{Spin}$ together with a twofold covering bundle mapping ξ defined by the following commutative diagram, analogous to (3.42):

$$\begin{tikzcd}
\mathbb{C}\mathrm{Spin}(M) \times \mathbb{C}\mathrm{Spin} \arrow[rr, "\xi \times \xi_o"] \arrow[d, "R"] & & \mathbb{C}L^{+}(M) \times \mathbb{C}L^{+} \arrow[d, "R"] \\
\mathbb{C}\mathrm{Spin}(M) \arrow[rr, "\xi"] \arrow[dr, "\pi"'] & & \mathbb{C}L^{+}(M) \arrow[dl, "\pi"] \\
& M &
\end{tikzcd} \tag{3.71}$$

Exactly as in (3.44) the *complex spinor bundle*, which we denote by the same symbol $S(M)$, is defined by

$$S(M) = \mathbb{C}\mathrm{Spin}(M) \times_{\zeta} \mathbb{C}^4 \,, \tag{3.72}$$

where ζ is the complex spin representation of (3.67).

Proceeding as on p. 41, any complex metric connection 1-form ω_m on $\mathbb{C}L^{+}(M)$ defines an unique complex spin connection ω_s via (3.45). Using a complex tetrad field as local cross section into $\mathbb{C}L^{+}(M)$, this complex spin connection defines precisely the same covariant spinor derivative as in (3.56), since the Lie algebra homomorphism $\zeta \circ \xi_o^{-1}$ in (3.69) has exactly the same structure as in (3.41). Because of this formal resemblance of the real and the complex spin geometry, we may speak of a "natural" extension of the real spin geometry to the complex case.[12]

[12]I could not find any textbook, where the complex extension of the spin geometry is discussed in such a great detail as here.

3.3 Fibre bundle background

3.3.1 Group structure

As it was outlined in the introduction to this chapter, we first construct a diagram of Lie group homomorphisms, which will then be copied into the framework of bundle mappings. Consider now the following diagram:

$$\begin{tikzcd} \mathbb{C}\mathrm{Spin} \times \mathbb{C}^{\times} \arrow[r, "\xi_o \times \mathrm{id}"] \arrow[d, "\zeta_c"'] & \mathbb{C}L^{+} \times \mathbb{C}^{\times} \arrow[r, "\theta_o"] & \boldsymbol{G} \arrow[r, "j_o"] & \mathrm{GL}(4, \mathbb{C}) \\ \mathrm{GL}(4, \mathbb{C}) & & & \end{tikzcd} \tag{3.73}$$

In the following, we shall explain the details of this diagram: First, the complex spin group $\mathbb{C}\mathrm{Spin}$ and the complex Lorentz group $\mathbb{C}L^{+}$, together with the spin mapping ξ_o, were defined in the foregoing section. The group of invertible elements of $\mathbb{C}$ is the abelian multiplicative group of non-zero complex numbers and is isomorphic to $\mathrm{GL}(1, \mathbb{C})$. It was denoted in the above diagram by

$$\mathbb{C}^{\times} := \mathbb{C} \setminus \{0\} \cong \mathrm{GL}(1, \mathbb{C}) \ . \tag{3.74}$$

If $\mathbb{C}^{\times}$ is restricted to unit elements, one gets of course U(1), which will become the electromagnetic gauge group later on. The reason, why $\mathbb{C}^{\times}$ instead of U(1) is considered here, is that a general complex linear connection might posses a trace part $\Gamma^{a}{}_{\mu a}$, which is not purely imaginary as in the field equation (2.27), and thus is not an U(1) potential, but a $\mathbb{C}^{\times}$ potential, see below (3.105). Note that we must explain the geometry of our extended spinor derivative in (2.14) *before* we take into account the field equations, since otherwise, the spinor derivative (2.14) and the Lagrangian $\mathcal{L}_m$ (2.15) based upon this derivative are not defined mathematically.

The representation ζ_c in (3.73) of the product group $\mathbb{C}\mathrm{Spin} \times \mathbb{C}^{\times}$ into $\mathrm{GL}(4, \mathbb{C})$ will be called the *extended spin representation* and will be needed below to construct the spinor bundle on the basis of the spin bundle. The representation ζ_c is defined in the following way,

$$\begin{aligned} \zeta_c : \mathbb{C}\mathrm{Spin} \times \mathbb{C}^{\times} &\longrightarrow \quad \mathrm{GL}(4, \mathbb{C}) \\ (\,(A, B)\,,\, c) &\longmapsto \zeta((A, B)) \cdot c^{-1} \ . \end{aligned} \tag{3.75}$$

Here we have written the complex spin group as $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$, see (3.60). The choice c^{-1} for the representation of $\mathbb{C}^{\times}$ is necessary in (3.75) in order to obtain the spinor derivative (2.14) and corresponds to the negative charge of the spinor. Other possible representations c^{ε}, $\varepsilon \in \mathbb{R}$, correspond to spinors with electric charge εe, see

below, (3.124). In subsesction 3.3.5, we will need the Lie algebra homomorphism of (3.75), which is simply given by

$$\zeta_c((\Lambda,\Lambda'),\,\lambda)) = \zeta((\Lambda,\Lambda')) - \lambda\mathbb{1}\,,\quad ((\Lambda,\Lambda'),\lambda)\in(\boldsymbol{sl}(2,\mathbb{C})\times\boldsymbol{sl}(2,\mathbb{C}))\times\mathbb{C}\,. \tag{3.76}$$

It remains to explain θ_o, $\boldsymbol{G}$, and j_o in the diagram (3.73). The Lie group homomorphism θ_o is defined by the following rule:

$$\begin{array}{rcl} \theta_o : \mathbb{C}L^+ \times \mathbb{C}^\times & \longrightarrow & \mathrm{GL}(4,\mathbb{C}) \\ (\Lambda\,,\,c) & \longmapsto & \Lambda c\,. \end{array} \tag{3.77}$$

The Lie group $\boldsymbol{G}$ is the image of θ_o,

$$\boldsymbol{G} := \theta_o(\mathbb{C}L^+\times\mathbb{C}^\times) = \{\Lambda c|\,\Lambda\in\mathbb{C}L^+,\, c\in\mathbb{C}^\times\}\,, \tag{3.78}$$

and j_o denotes the canonical inclusion of this group $\boldsymbol{G}$ into the full $\mathrm{GL}(4,\mathbb{C})$. Thus, by the definition of $\boldsymbol{G}$, θ_o in the diagram (3.73) is a surjective map. Moreover, it induces a Lie algebra isomorphism: The Lie algebra of $\mathbb{C}L^+\times\mathbb{C}^\times$ is clearly the cartesian product $\mathbb{C}\otimes\boldsymbol{l}\times\mathbb{C}$, where $\boldsymbol{l}$ is the Lie algebra of $L^+_\uparrow$ defined in (2.1). Let (A,λ) be an arbitrary element of this Lie algebra. Then it is mapped by θ_o (to be more precise, by its differential at the unit element $(\mathbb{1},1)$) to

$$\begin{aligned} \theta_o((A,l)) &= \theta_{o*}(A,\lambda) = \left.\frac{d}{dt}\right|_{t=0}\theta_o(\exp(t(A,\lambda))\,) \\ &= \left.\frac{d}{dt}\right|_{t=0}\theta_o(\exp(tA,t\lambda)\,) = \left.\frac{d}{dt}\right|_{t=0}\theta_o(\,(\exp(tA),\exp(t\lambda))\,) \\ &= \left.\frac{d}{dt}\right|_{t=0}[\,\exp(tA)\cdot\exp(t\lambda)] \\ &= \left.\frac{d}{dt}\right|_{t=0}[\,\exp(tA)\cdot 1 + \mathbb{1}\cdot\exp(t\lambda)] \\ &= A+\lambda\mathbb{1}\,. \end{aligned}$$

Since the elements of the Lorentz Lie algebra $\mathbb{C}\otimes\boldsymbol{l}$ do not contain any diagonal elements but only off-diagonal ones, the sum in the last line is direct.[13] Therefore, the Lie algebra of $\boldsymbol{G}$, denoted henceforth by $\boldsymbol{g}$, is the direct sum

$$\boldsymbol{g} = \mathbb{C}\otimes\boldsymbol{l}\oplus\mathbb{C}\mathbb{1}\,, \tag{3.79}$$

and θ_o is obviously an isomorphism between the two Lie algebras,

$$\theta_o : \mathbb{C}\otimes\boldsymbol{l}\times\mathbb{C} \xrightarrow{\cong} \mathbb{C}\otimes\boldsymbol{l}\oplus\mathbb{C}\mathbb{1}\,. \tag{3.80}$$

[13]If $A+\lambda\mathbb{1}=\lambda'\mathbb{1}$, then $A=0$, and if $A+\lambda\mathbb{1}=A'$, then $\lambda=0$.

This simple but subtle isomorphism property of θ_o will become crucial for the construction of the extended spin connection, see (3.92). We remark that, commonly, the Lie algebra of a product group such as $\mathbb{C}\otimes \boldsymbol{l}\times\mathbb{C}$ is already identified with $\mathbb{C}\otimes\boldsymbol{l}\oplus\mathbb{C}\mathbb{1}$. Thus, when the Lie algebra isomorphism θ_o is considered without its underlying Lie group mapping θ_o, (3.80) rather becomes a tautology.

We further remark that the Lie group homomorphism θ_o is a twofold map.[14]

3.3.2 Bundle structure

Having explained the basic group structure (3.73), we now construct its exact translation to the framework of fibre bundles. Thereby the Lie groups become the structure groups of principal bundles, and the Lie group homomorphisms become the accompanying group homomorphisms of bundle mappings (cf. 3.1.2).

The main fibre bundle structure of our theory can be summarized in the following "copy-diagram" of (3.73)

$$\begin{tikzcd}
(\mathbb{C}\mathrm{Spin}\times\mathbb{C}^\times)(M) \arrow[r, "\xi\times\mathrm{id}"] \arrow[d, "(*)"'] & (\mathbb{C}L^+\times\mathbb{C}^\times)(M) \arrow[r, "\theta"] & \boldsymbol{G}(M) \arrow[r, "j"] & F_c(M) \\
S_c(M) & & &
\end{tikzcd} \tag{3.81}$$

Let us first explain the various fibre bundles in this diagram: First, define the following trivial $\mathbb{C}^\times$ principal bundle $\mathbb{C}^\times(M)$ by

$$\mathbb{C}^\times(M) := M\times\mathbb{C}^\times\ . \tag{3.82}$$

Then $(\mathbb{C}\mathrm{Spin}\times\mathbb{C}^\times)(M)$ and $(\mathbb{C}L^+\times\mathbb{C}^\times)(M)$ are fibre product bundles of $\mathbb{C}\mathrm{Spin}(M)$ and $\mathbb{C}^\times(M)$, and of $\mathbb{C}L^+$ and $\mathbb{C}^\times(M)$, respectively, see 3.1.3. The bundle on the left-hand side of (3.81), $F_c(M)$, is the complex frame bundle defined on p. 9. The fibre bundle $\boldsymbol{G}(M)$ is a special subset of this complex frame bundle containing only tangent frames of the special form $(c\cdot {e_a}^\mu\partial_\mu)$. Here $({e_a}^\mu\partial_\mu)$ is a complex orthonormal frame of $\mathbb{C}L^+(M)$ and c is a non-zero complex number, thus,

$$\boldsymbol{G}(M) := \{(c\cdot {e_a}^\mu\partial_\mu)\mid ({e_a}^\mu\partial_\mu)\in\mathbb{C}L^+(M)\,,\ \ c\in\mathbb{C}^\times\}\ . \tag{3.83}$$

Then, $\boldsymbol{G}(M)$ is obviously a $\boldsymbol{G}$ principal bundle, where the right action of the group $\boldsymbol{G}$ is given by

$$(c\cdot {e_a}^\mu\partial_\mu)(\Lambda c') := (c'c\cdot {e_b}^\mu{\Lambda^b}_a\partial_\mu)\ . \tag{3.84}$$

[14]Due to the property of the Lorentz matrices we get from $\Lambda c=\Lambda' c'$ first the equality $c^2\eta = (\Lambda c)^T\eta(\Lambda c) = (\Lambda' c')^T\eta(\Lambda' c') = c'^2\eta \;\Leftrightarrow\; c'=\pm c$. This yields $\Lambda=\pm\Lambda'$ and thus $(\Lambda,c)=(\pm\Lambda',\pm c')$.

It is easy to show that this action is free and that the other axioms for the principal bundles in 3.1.1 are fulfilled.

The fibre bundle $S_c(M)$ at the bottom of (3.81) is not a principal bundle, but is the $\mathbb{C}^4$ vector bundle

$$S_c(M) := (\mathbb{C}\mathrm{Spin}\times\mathbb{C}^\times)(M) \times_{\zeta_c} \mathbb{C}^4 \tag{3.85}$$

associated to the product bundle $(\mathbb{C}\mathrm{Spin}\times\mathbb{C}^\times)(M)$ via the extended spin representation ζ_o in (3.75), see 3.1.4. We call $S_c(M)$ the *extended spinor bundle*.

We shall now explain the bundle mappings between the principal bundles of (3.81). Remembering that ξ in (3.81) denotes the complex spin structure as defined in (3.71), the bundle mapping $\xi \times \mathrm{id}$ is simply defined as follows: An element (u, v) of $(\mathbb{C}\mathrm{Spin}\times\mathbb{C}^\times)(M)$ is mapped to $(\xi(u), v)$ in $(\mathbb{C}L^+\times\mathbb{C}^\times)(M)$. Because of the trivial relation

$$(\xi\times\mathrm{id})(u\Lambda, vc) = (\xi(u)\xi_o(\Lambda), vc) = ((\xi\times\mathrm{id})(u,v))((\xi_o\times\mathrm{id})(\Lambda,c)) , \tag{3.86}$$

where $(\Lambda, c) \in \mathbb{C}\mathrm{Spin} \times \mathbb{C}^\times$, we see that $(\xi\times\mathrm{id}\, ,\, \xi_o\times\mathrm{id})$ is a bundle mapping as explained in 3.1.2.

To explain the bundle map θ, we denote an element of $(\mathbb{C}L^+ \times \mathbb{C}^\times)(M)$ by $(e_a{}^\mu\partial_\mu, c)$, where c is the $\mathbb{C}^\times$-component of the respective element in $\mathbb{C}^\times(M)$ over the same base point as the orthonormal frame $(e_a{}^\mu\partial_\mu)$. Note that such a simplified notation is possible here because $\mathbb{C}^\times(M)$ is a trivial bundle. Then, θ can be defined as follows

$$\begin{aligned} \theta : (\mathbb{C}L^+\times\mathbb{C}^\times)(M) &\longrightarrow\ G(M) \\ ((e_a{}^\mu\partial_\mu)\, ,\ c\,) &\longmapsto (c\, e_a{}^\mu\partial_\mu)\ . \end{aligned} \tag{3.87}$$

It is straightforward to show that (θ, θ_o) (cf. (3.77)) indeed defines a bundle mapping. Furthermore it is important at this point to note that the above construction of θ necessitates a trivial structure of the principal bundle $\mathbb{C}^\times(M)$, since otherwise there would be no well-defined multiplication of a tangent frame with a complex number. Since $\mathbb{C}^\times(M)$, when restricted to its U(1) subbundle, will constitute the electromagnetic U(1) bundle, see (3.130), we may say that the electromagnetic U(1) bundle is *necessarily trivial* in our theory.

Finally, the bundle map j in (3.81) is simply the canonical inclusion of $\boldsymbol{G}(M)$ into the frame bundle $F_c(M)$.

Let us briefly discuss the main feature of the bundle diagram (3.81): Our aim is to construct a covariant spinor derivative out of an arbitrary complex linear connection ω defined on the complex frame bundle $F_c(M)$. As outlined in 2.3.2 on p. 19, this can be done by pulling ω back onto an "intermediate bundle", which possesses a spin structure. In the following subsections, this will be realized with the help of the above diagram: We first pull ω back via j onto $\boldsymbol{G}(M)$, then via θ onto the product bundle $(\mathbb{C}L^+ \times \mathbb{C}^\times)(M)$, which possesses the spin structure $(\mathbb{C}\mathrm{Spin} \times \mathbb{C}^\times)(M)$.

Finally, if ω is further pulled back to this spin bundle via $\xi \times \mathrm{id}$, then it will define an unique covariant spinor derivative on the spinor bundle $S_c(M)$. The principal bundle $\boldsymbol{G}(M)$ located between the product bundle $(\mathbb{C}L^+ \times \mathbb{C}^\times)(M)$ and the frame bundle $F_c(M)$ is introduced in the diagram in order to make the pull-back procedure especially simple.

3.3.3 Extended spin connection

As has been said before, the fibre bundle diagram (3.81) will enable us to construct an unique spin connection on $(\mathbb{C}\mathrm{Spin}\times\mathbb{C}^\times)(M)$ starting from an arbitrary complex linear connection of the spacetime manifold by pulling back its connection 1-form on $F_c(M)$ along the horizontal line of the diagram from the right to the left.

To see that this procedure really works, let ω be an arbitrary connection 1-form on the complex frame bundle $F_c(M)$. The first step is to construct a connection on the bundle $\boldsymbol{G}(M)$. Since $\boldsymbol{G}(M)$ is a subbundle of $F_c(M)$, we may apply **Proposition 1** of 3.1.7. For the application, it is necessary to find a vector subspace $\boldsymbol{m}$ of $\boldsymbol{gl}(4, \mathbb{C})$, such that $\boldsymbol{gl}(4, \mathbb{C})$ is the direct sum of $\boldsymbol{m}$ and the Lie algebra $\boldsymbol{g}$ of $\boldsymbol{G}$ (3.79) having the additional property stated in that proposition. Define the vector subspace $\boldsymbol{m}$ by

$$\boldsymbol{m} := \{A \in \boldsymbol{gl}(4, \mathbb{C}) \mid A^T\eta - \eta A = 0 \ , \ \ \mathrm{Tr}(A) = 0\} \ . \tag{3.88}$$

It is straightforward to show that this $\boldsymbol{m}$ is indeed a $\mathbb{C}$-vector subspace of $\boldsymbol{gl}(4, \mathbb{C})$. Note that $\boldsymbol{m}$ is *not* a Lie subalgebra. Then, with the definition (3.79) of $\boldsymbol{g}$,

$$\boldsymbol{gl}(4, \mathbb{C}) = \boldsymbol{g} \oplus \boldsymbol{m} = \mathbb{C}\otimes\boldsymbol{l} \oplus \mathbb{C}\mathbb{1} \oplus \boldsymbol{m} \ . \tag{3.89}$$

To prove this assertion, we explicitly give the components of an element of $\boldsymbol{gl}(4, \mathbb{C})$ according to this decomposition,

$$\begin{array}{ccccccc} \boldsymbol{gl}(4, \mathbb{C}) & = & \mathbb{C}\otimes\boldsymbol{l} & \oplus & \mathbb{C}\mathbb{1} & \oplus & \boldsymbol{m} \\ A & = & \frac{1}{2}(A - \eta A^T\eta) & + & \frac{1}{4}\mathrm{Tr}A\cdot\mathbb{1} & + & \frac{1}{2}(A + \eta A^T\eta - \frac{1}{2}\mathrm{Tr}A\cdot\mathbb{1}) \end{array} \tag{3.90}$$

It is easy to show that the components given in (3.90) indeed fulfill the required algebraic properties. In order to employ **Proposition 1**, we must prove $(\Lambda c)\boldsymbol{m}(\Lambda c)^{-1} \subset \boldsymbol{m}$ for all $\Lambda c \in \boldsymbol{G}$. Using $\Lambda^T\eta\Lambda = \eta$, we have for an arbitrary element A of $\boldsymbol{m}$

$$\begin{aligned} \left(\Lambda c\, A\, (\Lambda c)^{-1}\right)^T \eta &= \Lambda^{-1\,T} A^T \Lambda^T \eta \ = \Lambda^{-1\,T} A^T \eta \Lambda^{-1} \\ &= -\Lambda^{-1\,T}\eta A\Lambda^{-1} = -\eta\Lambda A\Lambda^{-1} = -\eta\left(\Lambda c\, A\, (\Lambda c)^{-1}\right) \ , \\ \mathrm{Tr}\left(\Lambda c\, A\, (\Lambda c)^{-1}\right) &= \mathrm{Tr}A = 0 \ . \end{aligned}$$

With the help of **Proposition 1**, we now obtain a connection 1-form on $\boldsymbol{G}(M)$ by restricting ω to $\boldsymbol{G}(M)$ and taking its $\boldsymbol{g}$-component. This connection will be denoted by ω_G,

$$\omega_G := \boldsymbol{g}\text{-component of} \quad \omega|_{G(M)} \ . \tag{3.91}$$

As the next step, we construct a connection on the product bundle $(\mathbb{C}L^+ \times \mathbb{C}^\times)(M)$. Since the Lie algebra homomorphism of θ_o is actually an isomorphism, see (3.80), we may apply **Proposition 3** of 3.1.7 to the bundle map θ and obtain the following connection 1-form on $(\mathbb{C}L^+ \times \mathbb{C}^\times)(M)$

$$\omega_{lc} := \theta_o^{-1}\theta^*\omega_G \ . \tag{3.92}$$

Since $(\mathbb{C}L^+ \times \mathbb{C}^\times)(M)$ is a fibre product bundle, we can decompose this connection ω_{lc} by using **Proposition 2** of 3.1.7 as

$$\omega_{lc} = p^*\omega_l + q^*\omega_c \ , \tag{3.93}$$

where p and q are the canonical projections from $(\mathbb{C}L^+ \times \mathbb{C}^\times)(M)$ to $\mathbb{C}L^+(M)$ and $\mathbb{C}^\times(M)$, respectively, and ω_l and ω_c denote the connections on $\mathbb{C}L^+$ and $\mathbb{C}^\times(M)$ constructed canonically from ω_{lc}, see the proof of **Proposition 2**. Thus, ω_l is a complex Lorentz connection on $\mathbb{C}L^+$, whose algebraic components are given by the restriction of ω_{lc} to its $\mathbb{C} \otimes \boldsymbol{l}$-component, and ω_c is a $\mathbb{C}^\times$ potential on $\mathbb{C}^\times(M)$, whose algebraic component is the $\mathbb{C}^\times$-component of ω_{lc}.[15]

We can now construct the *extended spin connection* on $(\mathbb{C}\mathrm{Spin} \times \mathbb{C}^\times)(M)$ from ω_{lc} by using again **Proposition 3**, since the Lie algebra mapping $\xi_o \times \mathrm{id}$ (3.81) is an isomorphism, cf. (3.66). If we denote this spin connection by ω_{sc}, then

$$\begin{aligned} \omega_{sc} &= (\xi_o \times \mathrm{id})^{-1}(\xi \times \mathrm{id})^*\omega_{lc} & (3.94)\\ &= \xi_o^{-1}\xi^*p^*\omega_l + q^*\omega_c \ . & (3.95) \end{aligned}$$

3.3.4 Local cross sections

In order to obtain the components of the connection on the base manifold M from the various connection 1-forms introduced in the foregoing subsection we shall consider local cross sections of the principal bundles in the diagram (3.81).

[15] In the diploma thesis [Hor 94] these two connections ω_l and ω_c were assumed to be some *restrictions* of the full connection on $(\mathbb{C}L^+ \times \mathbb{C}^\times)(M)$ onto its "compontents" $\mathbb{C}L^+(M)$ and $\mathbb{C}^\times(M)$, see p. 67, above the formula (A.40). But this is not the correct way to express these two connections, since $\mathbb{C}L^+(M)$ and $\mathbb{C}^\times(M)$ might not be contained in $(\mathbb{C}L^+ \times \mathbb{C}^\times)(M)$ as natural subbundles, that is as "components". The problem is here, that there is no natural inclusion mapping from $\mathbb{C}^\times(M)$ into $(\mathbb{C}L^+ \times \mathbb{C}^\times)(M)$ if $\mathbb{C}L^+(M)$ is not a trivial bundle. Nevertheless, the formula (A.41) in the diploma thesis is formally correct, and can be used to decompose a linear connection into its metric part and its non-metric vector part.

Let $\mathcal{U}$ be an open subset of M, on which all the principal bundles considered so far admit local cross sections. Let

$$\sigma = (e_a{}^\mu \partial_\mu) \tag{3.96}$$

be a local cross section of the complex Lorentz bundle $\mathbb{C}L^+(M)$. Thus, σ is a complex orthonormal tetrad field. Although we could restrict our considerations only to the case of real tetrad fields as in chapter 2, we shall allow here for arbitrary complex tetrad fields, because we want to study the full *mathematical structure* of the bundle geometry without bothering about physics. As remarked on p. 9, complex tetrad fields are also allowed in our theory, if one does not consider the physical role of the tetrad fields themselves.

As in the case of real spin geometry, there exists a local cross section $\hat{\sigma}$ of the complex spin bundle $\mathbb{C}\mathrm{Spin}(M)$, such that the spin mapping ξ maps it onto σ, cf. (3.46),

$$\sigma = \xi(\hat{\sigma}) \ . \tag{3.97}$$

Since we want to consider cross sections of the product bundles $(\mathbb{C}\mathrm{Spin}\times\mathbb{C}^\times)(M)$ and $(\mathbb{C}L^+\times\mathbb{C}^\times)(M)$ in the diagram (3.81), we need a cross section of the principal bundle $\mathbb{C}^\times(M)$, which is merely a $\mathbb{C}^\times$-valued function, because $\mathbb{C}^\times(M)$ is a trivial bundle. At the moment, we choose a special function, denoted by $\hat{1}$, whose values equal constantly $1 \in \mathbb{C}^\times$,

$$\hat{1} : \mathcal{U} \longmapsto \mathbb{C}^\times, \quad p \longmapsto 1 \ . \tag{3.98}$$

Later on, we will consider arbitrary functions and elaborate the gauge transformations connected with the change from $\hat{1}$ to these functions.

Now, $(\sigma, \hat{1})$ and $(\hat{\sigma}, \hat{1})$ are clearly cross sections of the product bundles $(\mathbb{C}L^+\times\mathbb{C}^\times)(M)$ and $(\mathbb{C}\mathrm{Spin}\times\mathbb{C}^\times)(M)$, respectively. Remembering the definition of the bundle map θ in (3.87), we obtain the following commutative diagram of various cross sections constructed so far:

$$\begin{tikzcd}
(\mathbb{C}\mathrm{Spin}\times\mathbb{C}^\times)(M) \arrow[r, "\xi\times\mathrm{id}"] & (\mathbb{C}L^+\times\mathbb{C}^\times)(M) \arrow[r, "\theta"] & \boldsymbol{G}(M) \arrow[r, "j"] & F_c(M) \\
\mathcal{U} \arrow[r, "="] \arrow[u, "{(\hat{\sigma}, \hat{1})}"] & \mathcal{U} \arrow[r, "="] \arrow[u, "{(\sigma, \hat{1})}"] & \mathcal{U} \arrow[r, "="] \arrow[u, "1\cdot\sigma"] & \mathcal{U} \arrow[u, "1\cdot\sigma"]
\end{tikzcd} \tag{3.99}$$

3.3.5 Extended spinor derivative

Connections on the base space

Let ω be a connection 1-form on the complex frame bundle $F_c(M)$ and let

$$\Gamma^a{}_{\mu b} dx^\mu := ((1 \cdot \sigma)^* \omega)^a{}_b \tag{3.100}$$

be the $\boldsymbol{gl}(4, \mathbb{C})$-components of the pulled back connetion on the base space M. They are the anholonomic tetrad components of the general complex linear connection as introduced in (2.5). The superfluous factor 1 in front of σ is inserted here as well as in the diagram (3.99) in view of the U(1) gauge transformation considered in the next section.

Now consider the connection ω_G on $\boldsymbol{G}(M)$ defined in (3.91). If this connection is pulled back by the same local cross section $1 \cdot \sigma$ to M, then the resultant connection on the spacetime M will not be the same as in (3.100), but it will have only its $\boldsymbol{g}$-components. Thus, although the diagram (3.99) of the various cross sections is perfectly commutative, this property is lost when considering the connections, because the "mappings" between them, cf. equations (3.91) to (3.95), do not include only the mappings between the underlying topological spaces, but also various Lie algebra homomorphisms. Using the explicit decomposition (3.90), we take the $\boldsymbol{g}$-components of (3.100) to obtain

$$((1 \cdot \sigma)^* \omega_G)^a{}_b = \frac{1}{2}(\Gamma^a{}_{\mu b} - \Gamma_{b\mu}{}^a) dx^\mu \; + \; \frac{1}{4}\Gamma^c{}_{\mu c} \delta^a{}_b dx^\mu \; . \tag{3.101}$$

Next, the pull-back of ω_{lc} (3.92) via the cross section $(\sigma, \hat{1})$ in (3.99) results in the same expression,

$$\begin{aligned} ((\sigma, \hat{1})^* \omega_{lc})^a{}_b &= ((\sigma, \hat{1})^* \theta_o^{-1} \theta^* \omega_G)^a{}_b \\ &= (\theta_o^{-1} (\sigma \cdot \hat{1})^* \omega_G)^a{}_b \\ &= \frac{1}{2}(\Gamma^a{}_{\mu b} - \Gamma_{b\mu}{}^a) dx^\mu \; + \; \frac{1}{4}\Gamma^c{}_{\mu c} \delta^a{}_b dx^\mu \; , \end{aligned} \tag{3.102}$$

where we exploited the commutative rectangle at the centre of the diagram (3.99). Note that here the Lie algebra of $\mathbb{C}L^+ \times \mathbb{C}^\times$ has been trivially identified with $\mathbb{C} \otimes \boldsymbol{l} \oplus \mathbb{C}\mathbb{1}$, see the remark on p. 49. If we do not make such an identification, then the correct, but somewhat pedantic, expression reads

$$((\sigma, \hat{1})^* \omega_{lc})^a{}_b = (\frac{1}{2}(\Gamma^a{}_{\mu b} - \Gamma_{b\mu}{}^a) dx^\mu \; , \; \frac{1}{4}\Gamma^c{}_{\mu c} \delta^a{}_b dx^\mu) \; . \tag{3.103}$$

Of course, care must be taken when (3.101) and (3.102) are compared, since they belong to connection 1-forms on different principal bundles: Whereas in (3.101) the

plus-sign denotes merely an addition of different *algebraic* components, the plus-sign in (3.102) means the sum of two *geometrically* different connections, namely of (see (3.93))

$$(\sigma^* \omega_l)^a{}_b = \frac{1}{2}(\Gamma^a{}_{\mu b} - \Gamma_{b\mu}{}^a) dx^\mu \quad \text{and} \tag{3.104}$$

$$(\hat{1}^* \omega_c)^a{}_b = \frac{1}{4}\Gamma^c{}_{\mu c} \delta^a{}_b dx^\mu \,. \tag{3.105}$$

As we have said below (3.93), ω_l is a complex Lorentzian connection on $\mathbb{C}L^+(M)$, and ω_c is a $\mathbb{C}^\times$ potential on $\mathbb{C}^\times(M)$.

In a similar fashion, using the left commutative rectangle of (3.99) and the decomposition (3.95), we obtain the extended spin connection on the base space M,

$$\begin{aligned} (\hat{\sigma}, \hat{1})^* \omega_{sc} &= (\hat{\sigma}, \hat{1})^* (\xi_o^{-1} \xi^* p^* \omega_l + q^* \omega_c) \\ &= \xi_o^{-1} \sigma^* \omega_l \;+\; \hat{1}^* \omega_c \,. \end{aligned} \tag{3.106}$$

We now employ the extended spin representation ζ_c (3.75), its Lie algebra homomorphism (3.76), and equation (3.69) to obtain

$$\zeta_c((\hat{\sigma}, \hat{1})^* \omega_{sc}) = (\zeta \circ \xi_o^{-1}) \sigma^* \omega_l \;-\; \hat{1}^* \omega_c \tag{3.107}$$

$$= -\frac{1}{4}\gamma^b \gamma_a \cdot \frac{1}{2}(\Gamma^a{}_{\mu b} - \Gamma_{b\mu}{}^a) dx^\mu \;-\; \mathbb{1} \cdot \frac{1}{4}\Gamma^c{}_{\mu c} dx^\mu \tag{3.108}$$

$$= -\frac{1}{4}\Gamma_{a\mu b} \gamma^b \gamma^a dx^\mu \,. \tag{3.109}$$

The extended covariant spinor derivative

We are now able to construct the extended spinor derivative (2.14) on p. 11 by following analogous steps as in (3.53) to (3.56) for the construction fo the ordinary spinor derivative (2.12).

In the bundle diagram (3.81), Dirac spinors ψ are vector fields on the spinor bundle $S_c(M)$, which we represent as

$$\psi = [(\hat{\sigma}, \hat{1}) \,,\; \psi_{(\hat{\sigma}, \hat{1})}] \,, \tag{3.110}$$

where $\psi_{(\hat{\sigma}, \hat{1})}$ is a $\mathbb{C}^4$-valued function on $\mathcal{U}$. With the help of (3.109), the extended covariant spinor derivative reads

$$\begin{aligned} \nabla_\mu \psi &= [(\hat{\sigma}, \hat{1}) \,,\; \partial_\mu \psi_{(\hat{\sigma}, \hat{1})} + \zeta_c((\hat{\sigma}, \hat{1})^* \omega_{sc}(\partial_\mu)) \psi_{(\hat{\sigma}, \hat{1})}] \\ &= [(\hat{\sigma}, \hat{1}) \,,\; \partial_\mu \psi_{(\hat{\sigma}, \hat{1})} - \frac{1}{4}\Gamma_{a\mu b} \gamma^b \gamma^a \psi_{(\hat{\sigma}, \hat{1})}] \,, \end{aligned} \tag{3.111}$$

which may be written simply as

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{4}\Gamma_{a\mu b} \gamma^b \gamma^a \psi \,. \tag{3.112}$$

This is precisely our extended covariant spinor derivative (2.14).

Decomposition principle

We shall now turn our attention to the mathematical structure of the connection (2.27), which is the result of the field equation (2.20):

$$\Gamma^a{}_{\mu b} = \widehat{\Gamma}^a{}_{\mu b} + \delta^a{}_b \cdot S_\mu \ , \tag{3.113}$$

where $\widehat{\Gamma}^a{}_{\mu b}$ is a complex Lorentzian connection compatible with the metric, $\widehat{\Gamma}_{a\mu b} = -\widehat{\Gamma}_{b\mu a}$. If we insert this connection (2.27) into the above formulae (3.104) and (3.105), then

$$(\sigma^*\omega_l)^a{}_b = \widehat{\Gamma}^a{}_{\mu b} dx^\mu \ , \tag{3.114}$$
$$(\hat{1}^*\omega_c)^a{}_b = \delta^a{}_b \cdot S_\mu dx^\mu \ . \tag{3.115}$$

Thus, we can uniquely decompose the resultant connection in (2.27) in accordance with (3.93), that is, as the sum of a complex Lorentzian connection on $\mathbb{C}L^+(M)$ and a $\mathbb{C}^\times$ potential on $\mathbb{C}^\times(M)$. In so doing, we of course interpret the connection (2.27) as a connection resulting from the product bundle $(\mathbb{C}L^+ \times \mathbb{C}^\times)(M)$ and not from the frame bundle $F_c(M)$. This point of view can only be taken after the field equations for the connection have been considered, but not before, since an arbitrary linear connection does not possess the special structure of (2.27).

We now discuss the extended spinor derivative (2.14),

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{4}\Gamma_{a\mu b}\sigma^{ba}\psi - \frac{1}{4}\Gamma^c{}_{\mu c}\psi \ . \tag{3.116}$$

We see that this extended spinor derivative is already decomposed *formally* into two contributions $-\frac{1}{4}\Gamma_{a\mu b}\sigma^{ba}$ and $-\frac{1}{4}\Gamma^c{}_{\mu c}$. But now regarding the equations (3.106) to (3.109) it is clear that this decomposition is based on a geometric foundation: The extended spin connection ω_{sc} is indeed the sum of two different connections, namely the "ordinary" complex spin connection ω_s defined by $\omega_s = \xi_o^{-1}\xi^* p^* \omega_l$ in equation (3.95), cf. 3.2.4, and a $\mathbb{C}^\times$ potential ω_c on $\mathbb{C}^\times(M)$, see (3.95). Whereas ω_s gives rise to the covariant derivative characterized by

$$\zeta_c(\hat{\sigma}^*\omega_s) = \zeta_c(\hat{\sigma}^*(\xi_o^{-1}\xi^* p^*\omega_l)) = (\zeta_c\xi_o^{-1})\hat{\sigma}^*\omega_l = -\frac{1}{4}\Gamma_{a\mu b}\sigma^{ab}dx^\mu \ , \tag{3.117}$$

the $\mathbb{C}^\times$ potential ω_c leads to

$$-\,\hat{1}^*\omega_c = -\frac{1}{4}\Gamma^c{}_{\mu c}dx^\mu \ , \tag{3.118}$$

cf. (3.106) to (3.109). This decomposition of the spinor derivative is, contrary to the decomposition of the linear connection as considered in (3.114) and (3.115), valid already before the field equations have been taken into account. This property of

the extended spinor derivative (2.14) is indeed necessary for the construction of the basic matter Lagrangian $\mathcal{L}_m$ (2.15), as was said in the discussion following (3.74).

Note that the factor $\frac{1}{4}$ in front of the trace $\Gamma^c{}_{\mu c}$ in (3.108) has its real origin in the algebraic decomposition (3.90), whereas in (2.14), this factor seems to be caused by the overall factor 1/4 of the usual covariant spinor derivative (2.12).

3.4 Electromagnetic gauge transformation

Let us now study the $\mathbb{C}^\times$ gauge transformation, which, if restricted to U(1), will become the electromagnetic phase transformation. Let λ be a $\mathbb{C}$-valued function on $\mathcal{U}$ and let

$$\hat{\lambda} := \hat{1} \cdot \exp(\lambda) \tag{3.119}$$

be an arbitrary $\mathbb{C}^\times$-valued function on $\mathcal{U}$, viewed as a cross section of $\mathbb{C}^\times(M)$. Then, according to the gauge transformation law (3.27), we obtain for the $\mathbb{C}^\times$ connection ω_c

$$\begin{aligned} \hat{\lambda}^*\omega_c(\partial_\mu) &= e^{-\lambda}\hat{1}^*\omega_c(\partial_\mu)e^\lambda + e^{-\lambda}\partial_\mu e^\lambda \ = \ \hat{1}^*\omega_c(\partial_\mu) + \partial_\mu\lambda \\ &= \frac{1}{4}\Gamma^c{}_{\mu c} + \partial_\mu\lambda \ , \end{aligned} \tag{3.120}$$

which should be compared with the expression (3.105).[16] This is the $\mathbb{C}^\times$ gauge transformation of the $\mathbb{C}^\times$ potential $\frac{1}{4}\Gamma^c{}_{\mu c}$. Since this transformation affects only quantities defined on or derived from the principal bundle $\mathbb{C}^\times(M)$, all other quantities on the complex Lorentz bundle $\mathbb{C}L^+(M)$ or on the complex spin bundle $\mathbb{C}\mathrm{Spin}(M)$ remain unchanged. Thus, especially, the complex Lorentzian connection (3.104) and the complex tetrad field σ remain fixed. This would not hold true any longer if, in the diagram (3.99), the complex frame bundle $F_c(M)$ or $G(M)$ are considered, see below.

To study the gauge transformation of the Dirac spinor ψ, we use the gauge transformation property (3.17) of the vector fields to get (cf. (3.110) and (3.75))

$$\begin{aligned} \psi \quad &= [(\hat{\sigma}, \hat{1})\, , \psi_{(\hat{\sigma},\hat{1})}] \ = \ [(\hat{\sigma}, \hat{\lambda})\, , \psi_{(\hat{\sigma},\hat{\lambda})}]\ , \\ \psi_{(\hat{\sigma},\hat{\lambda})} &= \zeta_c(((\mathbb{1},\mathbb{1}),e^\lambda)^{-1})\psi_{(\hat{\sigma},\hat{1})} \ = \ e^\lambda\psi_{(\hat{\sigma},\hat{1})}\ . \end{aligned} \tag{3.121}$$

In summary, the $\mathbb{C}^\times$ gauge transformation reads as follows:

$$e_a{}^\mu \mapsto e_a{}^\mu, \quad \frac{1}{4}\Gamma^c{}_{\mu c} \mapsto \frac{1}{4}\Gamma^c{}_{\mu c} + \partial_\mu\lambda, \quad \psi \mapsto e^\lambda\psi\ . \tag{3.122}$$

[16]Note that $\hat{\lambda}^*\omega_c = \hat{\lambda}^*\omega_c(\partial_\mu)\cdot dx^\mu$.

3.4.1 Further extension of the spinor derivative

In the discussion following (2.14), we remarked that the extension of the spinor derivative was not unique. In 2.4 we have exploited the remaining ambiguity to further extend the spinor derivative. To obtain the most general spinor derivative (2.56d), only a slight change of the extended spin representation ζ_c (3.75) is necessary. We now define

$$\begin{aligned} \zeta_\varepsilon : \mathbb{C}\mathrm{Spin} \times \mathbb{C}^\times &\longrightarrow \quad \mathrm{GL}(4, \mathbb{C}) \\ (\,(A,B)\,,\,c) &\longmapsto \zeta((A,B))\cdot c^\varepsilon\,, \end{aligned} \tag{3.123}$$

where $\varepsilon \in \mathbb{R}$. Using this spin representation, it is easy to show that Dirac spinors now transform as (cf. (3.121))

$$\psi_{(\hat\sigma,\hat\lambda)} = \zeta_\varepsilon(((\mathbb{1},\mathbb{1}),e^\lambda)^{-1}\psi_{(\hat\sigma,\hat 1)} = e^{-\varepsilon\lambda}\psi_{(\hat\sigma,\hat 1)}\ . \tag{3.124}$$

As in (3.107), we can compute the spin connection corresponding to ζ_ε using an arbitrary cross section $(\hat\sigma, \hat\lambda)$ to give the following result:

$$\begin{aligned} \zeta_\varepsilon((\hat\sigma,\hat\lambda)^*\omega_{sc}) &= (\zeta\circ\xi_o^{-1})\sigma^*\omega_l \ + \ \varepsilon\cdot\hat\lambda^*\omega_c \\ &= -\frac{1}{4}\gamma^b\gamma_a\cdot\frac{1}{2}({\Gamma^a}_{\mu b} - {\Gamma_{b\mu}}^a)dx^\mu \ + \ \varepsilon(\frac{1}{4}{\Gamma^c}_{\mu c} + \partial_\mu\lambda)dx^\mu\ . \end{aligned} \tag{3.125}$$

3.4.2 Restriction to U(1)

So far we have dealt with the group $\mathbb{C}^\times$, which was needed to construct the bundle structure (3.81). In order to restrict $\mathbb{C}^\times$ to its subgroup U(1), we first observe that the adjoint spinor[17] transforms under the $\mathbb{C}^\times$ gauge transformation (3.121) according to

$$\overline{\psi}_{(\hat\sigma,\hat\lambda)} = (\psi_{(\hat\sigma,\hat\lambda)})^\dagger\gamma^0 = e^{\overline{\lambda}}\,\overline{\psi}_{(\hat\sigma,\hat 1)}\ , \tag{3.126}$$

where $\overline{\lambda}$ means the complex conjugate of λ. Due to the "covariance" of the covariant derivative in the local representation, see the second line of (3.29), we have

$$\nabla_\mu\psi_{(\hat\sigma,\hat\lambda)} = e^\lambda\nabla_\mu\psi_{(\hat\sigma,\hat 1)}\ , \tag{3.127}$$

so that the matter Lagrangian $\mathcal{L}_m$ in (2.15) does not remain invariant under the whole $\mathbb{C}^\times$ gauge transformation, but changes

$$\mathcal{L}_m \longmapsto \exp(\overline{\lambda}+\lambda)\mathcal{L}_m\ . \tag{3.128}$$

[17] Adjoint spinors can be defined analogously to spinors as vector fields on an associated vector bundle of the extended spin bundle $(\mathbb{C}\mathrm{Spin}\times\mathbb{C}^\times)(M)$, where the representation of the extended spin group $\mathbb{C}\mathrm{Spin}\times\mathbb{C}^\times$ in $\mathrm{GL}(4,\mathbb{C})$ is taken to be the adjoint representation $(\gamma^0\zeta_c^\dagger\gamma^0)^T$. However, to avoid too much congestion in the exposition, we prefer to represent adjoint spinors only locally by simply taking the adjoint of an ordinary spinor.

Since we *have to* demand the invariance of the Lagrangian, we conclude

$$\begin{aligned} \overline{\lambda} + \lambda &= 0 \quad \text{for all} \quad \lambda \quad \Longrightarrow \\ \exp(\lambda) &\in \mathrm{U}(1) \ , \end{aligned} \tag{3.129}$$

so that we must not consider the whole group $\mathbb{C}^\times$, but only its subgroup U(1) of unit elements. As a consequence, instead of $\mathbb{C}^\times(M)$, its reduced bundle $\mathrm{U}(1)(M)$ must be considered. So, throughout this chapter, we subsequently

$$\text{replace every } \mathbb{C}^\times \text{ by } \mathrm{U}(1). \tag{3.130}$$

3.4.3 Further properties of the gauge transformation

In (3.120), we discussed the gauge transformation of the $\mathbb{C}^\times$, or, because of (3.130), of the U(1) potential ω_c on $\mathrm{U}(1)(M)$. We now want to study the same gauge transformation on the complex frame bundle $F_c(M)$. If we replace in the diagram (3.99) the trivial cross section $\hat{1}$ by $\hat{\lambda}$ defined in (3.119), we see that the cross section of the frame bundle becomes $e^\lambda \sigma$. The pull-back of the complex linear connection 1-form ω via this cross section does not remain unchanged, but transforms according to

$$((e^\lambda \sigma)^* \omega(\partial_\mu))^a{}_b = e^{-\lambda} (\sigma^* \omega(\partial_\mu))^a{}_b e^\lambda + \delta^a{}_b e^{-\lambda} \partial_\mu e^\lambda = \Gamma^a{}_{\mu b} + \delta^a{}_b \partial_\mu \lambda \ . \tag{3.131}$$

Thus, the connection trace $\Gamma^c{}_{\mu c}$ still transforms in a similar manner as in (3.120). But now the cross section $e^\lambda \sigma$ is no longer an *orthonormal* tetrad field, but only *orthogonal*. So, unlike σ, this cross section can not be "lifted" to a cross section of the spin bundle, and, therefore, no local representation of Dirac spinors (cf. (3.121)) can be defined for $e^\lambda \sigma$. Even worse, any tangent vector X, written in the tetrad components $X = X^a (e_a{}^\mu \partial_\mu)$, becomes now charged, since X^a is transformed to $e^{-\lambda} X^a$ due to the gauge transformation rule

$$X = X^a (e_a{}^\mu \partial_\mu) = e^{-\lambda} X^a (e^\lambda e_a{}^\mu \partial_\mu) \ . \tag{3.132}$$

For these mathematical and physical reasons, it is *not allowed* to consider the U(1) gauge transformation on the frame bundle $F_c(M)$ or, equivalently, on $\boldsymbol{G}(M)$, but only on the product bundles in the diagram (3.99).

Stated differently, we must discard the right and the middle commutative rectangles in the diagram (3.99) and retain only the left rectangle. In this way, we detach the U(1) potential ω_c and its U(1) gauge transformation completely from the basic complex linear connection ω and also from the basic frame bundle geometry of the spacetime manifold M.

3.4.4 Gauging the torsion trace

Suppose now that we do *not* detach ω_c from ω but consider (3.131) as the true U(1) gauge transformation on $F_c(M)$, aiming at a gauge transformation of the torsion trace T_μ. In order to calculate the torsion trace from the transformed connection (3.131) we ask about its coordinate components.

Denoting the local coordinate frame by

$$k := (\partial_\mu) = (\frac{\partial}{\partial x^\mu}) \ , \tag{3.133}$$

we reexpress it in terms of the transformed cross section $e^\lambda \sigma$ used to obtain (3.131),

$$\begin{aligned} k \quad &= (e^\lambda \sigma) \cdot (e^{-\lambda} \Lambda) \quad \Leftrightarrow \\ (\partial_\mu) &= (e^\lambda e_a{}^\nu \partial_\nu) \cdot (e^{-\lambda} e^a{}_\mu) \ , \end{aligned} \tag{3.134}$$

where the expression $(e^{-\lambda} e^a{}_\mu)$ containing the reciprocal tetrad $e^a{}_\mu$ plays the role of the gauge transforming matrix Λ, compare with (3.17). With the help of the gauge transformation law (3.27) we obtain the desired coordinate components of (3.131):

$$\begin{aligned} (k^* \omega(\partial_\mu))^\alpha{}_\beta &= (e^\lambda \Lambda^{-1} \cdot (e^\lambda \sigma)^* \omega(\partial_\mu) \cdot e^{-\lambda} \Lambda + (e^\lambda \Lambda^{-1}) \cdot \partial_\mu (e^{-\lambda} \Lambda))^\alpha{}_\beta \\ &= e_a{}^\alpha (\Gamma^a{}_{\mu b} + \delta^a{}_b \partial_\mu \lambda) e^b{}_\beta + \delta^\alpha{}_\beta \partial_\mu (-\lambda) + e_c{}^\alpha \partial_\mu e^c{}_\beta \\ &= e_a{}^\alpha \Gamma^a{}_{\mu b} e^b{}_\beta + e_c{}^\alpha \partial_\mu e^c{}_\beta \\ &= \Gamma^\alpha{}_{\mu\beta} \ . \end{aligned} \tag{3.135}$$

This result is totally independent from the U(1) gauge function λ. Thus, we obtain the familiar result that the torsion trace T_μ can not be gauged with U(1).

Despite this undoubted result some authors like McKellar [McK 79] and Borchsenius [Bor 76a] regarded the so-called λ-transformation, first introduced by Einstein [Ein 55], as the U(1) gauge transformation for the torsion trace. This λ-gauge transformation reads

$$\Gamma^\alpha{}_{\mu\beta} \longmapsto \Gamma^\alpha{}_{\mu\beta} + \delta^\alpha{}_\beta \partial_\mu \lambda \ , \tag{3.136}$$

where λ is now an arbitrary complex valued function on the spacetime manifold M. It was introduced from the purely formal reason, that the Ricci-scalar R (2.10c) remains invariant under (3.136). One might ask, if there is any sensible way to understand (3.136) as a geometric feature?

One suggestion might be to regard it as part of a conformal transformation of the coordinate reference frame k (3.133), that is,

$$k = (\partial_\mu) \longmapsto (e^\lambda \partial_\mu) \ , \tag{3.137}$$

in analogy to the transformation of the tetrad field $\sigma \mapsto e^\lambda \sigma$. One can easily see, that this indeed results in the λ-transformation (3.136) of the connection by using (3.27).

But the problem is that $(e^\lambda \partial_\mu)$ is no longer a coordinate reference frame:[18] Now the components of the connection on the right-hand side in (3.136) are no longer coordinate components, forbidding their use for the ordinary covariant derivative ∇_μ in coordinate components. Instead, everything must now be represented in the special frame $(e^\lambda \partial_\mu)$. For example, a vector field X^μ in coordinate components would now read $e^{-\lambda} X^\mu$, so that, if (3.137) is regarded as the electromagnetic phase transformation, every covariant vector field would be charged. This situation is analogous to (3.132).

Thus, it seems that there is no sound way to get a gauged torsion vector T_μ. We repeat, that the only way out of this problem is to detach the U(1) potential ω_c completely from the frame bundle geometry by considering only the left rectangle in the diagram (3.99). In this way, we obtain a consistent U(1) gauge theory of electromagnetism and are able to interpret the vector $S_\mu = \frac{1}{4}\Gamma^a{}_{\mu a}$ as the true electromagnetic potential via (2.47). The torsion vector T_μ is related to the potential A_μ only formally, as explained in 2.3.3 on p. 21.

[18] This can be easily verified, since $[e^\lambda \partial_\mu, e^\lambda \partial_\nu] \neq 0$ unless λ is constant everywhere.

Chapter 4

Spin-Spin Contact Interaction

One interesting consequence of the Einstein–Cartan theory is the prediction of a contact interaction between spinning particles. In the introduction, we have briefly discussed the case of Dirac particles, see (1.7) and (1.8). Since the contact interaction is coupled to the square[1] of the Planck length l_0^2, it is hopelessly too small to be detected in laboratory experiments [Sto 85].

However, at high matter densities in the early universe, this tiny interaction can become even stronger than the mass effects of the interacting particles, see e.g. [Heh 73, Heh 76]. And, as was remarked by Kanno [Kan 88], at the high temperature predominant in this early epoch, the contact interaction becomes much stronger than the weak interaction: At a first glance, the contact interaction in (1.7) seems to be only a certain copy of the weak interaction, when this last interaction is written in the phenomenological Fermi contact form, i.e. without the gauge bosons. Since the Fermi coupling constant of the weak interaction is about $1.2 \times 10^{-5}\text{GeV}^{-2}$, whereas the constant of the contact interaction is of the order 10^{-37}GeV^{-2}, one may conclude that it does not make sense to look for an observable effect of the contact interaction in the presence of the weak interaction. However, it is well-known that the standard model of the electroweak interaction posseses a phase transition, where the broken symmetry is restored above a critical temperature of 100GeV, see e.g. [Kir 72, Dol 74, Din 92]. Above this temperature, the weak interaction becomes a long-range interaction of equal strength as the electromagnetic interaction, and the current-current terms are no longer appropriate to describe the electroweak forces. On the other hand, the contact interaction term in the Einstein–Cartan theory persists regardless of the energy scale considered, since it is directly induced by torsion without any intermediating bosons.[2]

In the early universe, when the density of spinning particles exceeded some crit-

[1] Note that in (1.7) we have $l_0^4/k = l_0^2 \times \hbar c$.

[2] If the energy scale is as high as the quantum gravity scale, then this remark may become incorrect, since then the geometry of spacetime (including torsion) must be quantized.

ical value, the contact interaction also leads to pair creations, see [Ker 75, Rum 79]. As was noted by Kerlick [Ker 75], the required mass density is more than thirty orders of magnitude smaller than the density required for pair creation via tidal forces caused by the curvature of spacetime [Zel 70]. Thus, the torsion-induced pair creation effects are much stronger and more likely than the curvature effects, and must be taken into account in the discussion of the scenario of the early universe [Ker 75].

The contact interaction might also influence the singularity behaviour [Haw 73] of the universe. Whereas Kerlick [Ker 75] concluded that torsion enhances singularity, other authors came to the opposite conclusion, namely that the contact interaction prevents it, see e.g. [Heh 74, Kuc 78, Nur 83, Kan 88].

We may say that the torsion-induced contact interaction has important consequences on the early stage of the universe. But so far, no prediction has been made which can be investigated by present astronomical observations. One reason for the uncertainty of the predictions is, of course, that the spin-spin contact interaction is very weak and takes place only in a small time interval during the early epoch of the universe. Another reason might be that quantum field theoretic investigations have been completely left out in most cases (see however [Kan 88] and [Gvo 85]). One reason for the omission of quantization is that Einstein–Cartan theory, like other gravitational theories, can not be quantized rigorously, that is, in a renormalizable way. Therefore, any quantization of the contact interaction is necessarily incomplete as physical theory.

In this chapter, we shall try to step towards a more realistic view of the spin-spin contact interaction by quantizing it in the first Born approximation.

First of all we must find such a contact interaction in our theory developed in chapter 2. This is done by considering a many-particle theory. It turns out that the resulting spin-spin interaction differs from the one of the Einstein–Cartan theory in not containing any self-interactions of fermions.

In the next section we discuss the works of Kerlick [Ker 75] and of Rumpf [Rum 79]. These authors studied the shift of the energy spectrum of a Dirac particle due to a constant background torsion field. They both concluded that the contact interaction is attractive for the opposite spin direction of interacting fields, but repulsive for aligned spins, and that it does not depend on whether one considers matter or anti-matter; thus, one may speak of a "universal" interaction [Ker 75]. Here we will apply these considerations to the contact interaction of our theory. The resulting energy shifts differ significantly from the results of the Einstein–Cartan theory.

In the third section, we investigate the new spin-spin contact interaction as well as the ordinary contact interaction of the Einstein–Cartan theory by quantizing both interactions in the first Born approximation. As a result, neither interaction is "universal" as first proposed by Kerlick in [Ker 75] for the ordinary Einstein–Cartan

contact interaction.

4.1 Many-particle theory

4.1.1 The missing contact interaction

In Einstein–Cartan theory the Lorentzian connection (1.5) is influenced by spinning particles. It possesses a non-vanishing contorsion part built from the axial current $\overline{\psi}\gamma^5\gamma^\mu\psi$, see (1.6). This contribution of Dirac fields to geometry results in the characteristic spin-spin contact interaction in the energy-momentum equation (1.7) as well as in the Dirac equation (1.8).

In chapter 2 we have seen that the resultant connection (2.27) also contains a non-vanishing contorsion, now built from both vector and axial currents. But there we could not observe a spin-spin contact interaction like in the Einstein–Cartan theory. Neither the energy-momentum equation (2.46d) nor the Dirac equation (2.46a) contain contact interaction terms, this being in contrast to the Einstein–Cartan theory.

But this does not mean that there is no contact interaction at all. The reason for the absence of the contact interaction is that so far we have treated a classical single particle field theory: In the Dirac equation (2.36) the cubic self-interaction term

$$(j_\mu + j^5_\mu)\gamma^\mu\psi = 0 \tag{4.1}$$

vanished due to the identity (2.38). Now, let us consider the basic Dirac equation (2.34),

$$i\gamma^\mu(\nabla^*_\mu - S_\mu)\psi - \frac{mc}{\hbar}\psi + (\frac{3}{2}iU_\mu + \frac{1}{8}V_\mu\gamma^5)\gamma^\mu\psi = 0\ , \tag{4.2}$$

which is valid without refering to the field equations for the connection, but uses only the 4-vector decomposition (2.19). The last term containing the vectors U_μ and V_μ vanishes due to the field equation (2.24) and (2.38). Now, if these two vectors have not only contributions from the same Dirac field ψ, but also from some other, different, Dirac field, say χ, so that for example $U_\mu = -il_0^2/4(\overline{\psi}\gamma_\mu\psi + \overline{\chi}\gamma_\mu\chi)$, then we would obtain

$$(\overline{\chi}\gamma_\mu\chi + \overline{\chi}\gamma^5\gamma_\mu\chi\gamma^5)\gamma^\mu\psi \neq 0 \tag{4.3}$$

instead of (4.1) in the Dirac equation (4.2).

Therefore, in order to observe the missing spin-spin contact interaction in our theory, we must consider a many-particle system.

4.1.2 Many-particle system

To discover the spin-spin contact interaction we discuss a many-particle system consisting of spinors ψ_z with charges $\varepsilon(z)e$, $\varepsilon(z) \in \mathbb{R}$, and masses m_z, where z is a counting index. In (2.15) only the matter Lagrangian $\mathcal{L}_m$ changes. This Lagrangian now becomes a sum of Lagrangians for each spinor ψ_z, its spinor derivative given by (2.56d) with $\varepsilon = \varepsilon(z)$, thus,

$$\begin{aligned} \mathcal{L} = g \cdot \hbar c \sum_z \, [i\overline{\psi}_z \gamma^\mu (\partial_\mu - \frac{1}{4}\Gamma_{a\mu b}\sigma^{ba} + \frac{\varepsilon(z)}{4}\Gamma^a{}_{\mu a})\psi_z - \frac{m_z c}{\hbar}\overline{\psi}_z \psi_z] \\ - \frac{g}{2k}R + \frac{g}{4k} l^2 Y_{\mu\nu}Y^{\mu\nu} \ . \end{aligned} \tag{4.4}$$

Instead of the field equation (2.21), we now obtain quite analogously

$$\begin{aligned} 0 &= \frac{\delta}{\delta \Gamma^a{}_{\mu b}}(\mathcal{L}_m + \mathcal{L}_G + \mathcal{L}_Y) \cdot \delta^\gamma{}_\mu e^{a\alpha} e_b{}^\beta \cdot \frac{k}{g} \\ &= \ \frac{1}{4} i l_0^2 \sum_z \, [\overline{\psi}_z \gamma^\gamma \sigma^{\beta\alpha}\psi_z + \frac{\varepsilon}{4}\overline{\psi}_z \gamma^\gamma \psi_z g^{\alpha\beta}] \\ &\quad - \frac{1}{2}[\Sigma^{\beta\epsilon}{}_\epsilon g^{\alpha\gamma} + \Sigma^{\epsilon}{}_{\epsilon}{}^{\alpha} g^{\gamma\beta} - \Sigma^{\beta\alpha\gamma} - \Sigma^{\gamma\beta\alpha}] \\ &\quad - l^2 g^{\alpha\beta}\nabla^*_\nu Y^{\nu\gamma} \ . \end{aligned} \tag{4.5}$$

This equation can be handled just in the same way as in the discussion following (2.21) by using the 4-vector decomposition and contraction techniques. For example, if we use the expression (2.22) and contract (4.5) with $g_{\alpha\beta}$, then we obtain now

$$0 = i l_0^2 \sum_z \varepsilon(z)\overline{\psi}_z \gamma^\gamma \psi_z - 4 l^2 \nabla^*_\nu Y^{\nu\gamma} \tag{4.6}$$

instead of (2.23c). In a similar fashion, we get for the vectors U^α, Q^α and V_δ

$$-Q^\alpha = U^\alpha = -i l_0^2/4 \sum_z \overline{\psi}_z \gamma^\alpha \psi_z \quad \text{and} \quad V_\delta = 3 l_0^2 \sum_z \overline{\psi}_z \gamma^5 \gamma_\delta \psi_z \ , \tag{4.7}$$

to be compared with (2.24). The resultant connection is formally the same as in (2.27), but the vector and axial currents occuring in (2.28) have to be replaced by the sums of individual currents via (4.7), thus,

$$\Gamma^a{}_{\mu b} \ = \ \widehat{\Gamma}^a{}_{\mu b} + \delta^a{}_b S_\mu \ , \tag{4.8a}$$

$$\widehat{\Gamma}^a{}_{\mu b} \ = \ \{{}^a{}_{\mu b}\} + \frac{1}{4} l_0^2 \sum_z \, (i\overline{\psi}_z \gamma^a \psi e_{b\mu} - i e^a{}_\mu \overline{\psi}_z \gamma_b \psi - \eta^a{}_{\mu b d}\overline{\psi}_z \gamma^5 \gamma^d \psi_z) \ . \tag{4.8b}$$

We remark, that (4.6) is the correct inhomogeneous Maxwell equation for the many-particle theory: In view of (2.26), (2.47) and (2.49), we can rewrite it as

$$\sum_z \varepsilon(z) e \cdot \overline{\psi}_z \gamma^\gamma \psi_z = \nabla^*_\nu F^{\nu\gamma} \ . \tag{4.9}$$

Let us now discuss the Dirac equation. It is easy to see that the Dirac equation (2.34) suffers minor changes only,

$$i\gamma^\mu(\nabla^*_\mu + \varepsilon(z)S_\mu)\psi_z - \frac{m_z c}{\hbar}\psi_z + (\frac{3}{2}iU_\mu + \frac{1}{8}V_\mu\gamma^5)\gamma^\mu\psi_z = 0\ , \tag{4.10}$$

where the vectors U_μ and V_μ are now given by (4.7). Thus, with the help of the identity (2.38), this can be reexpressed as

$$i\gamma^\mu(\nabla^*_\mu + \varepsilon(z)S_\mu)\psi_z - \frac{m_z c}{\hbar}\psi_z - \frac{3}{8}l_0^2 \sum_{z'\neq z}(\overline{\psi}_{z'}\gamma_\mu\psi_{z'} + \overline{\psi}_{z'}\gamma^5\gamma_\mu\psi_{z'}\gamma^5)\gamma^\mu\psi_z = 0\ . \tag{4.11}$$

This Dirac equation contains clearly a spin-spin contact interaction, which, however, differs from the interaction in the Einstein–Cartan theory, cf. (1.8). The interaction term in the Dirac equation (4.11) contains besides the axial currents also the vector currents and allows therefore only interactions between distinct particles. So, at least on the classical level, both contact interactions differ significantly.

The field equation for energy-momentum (2.43) gains a new spin-spin interaction term $W_{\alpha\beta}$ on the right-hand side,[3]

$$\begin{aligned} W_{\alpha\beta} &= \frac{3}{k}(-U_\mu U^\mu + \frac{1}{12^2}V_\mu V^\mu) && \text{(4.12a)}\\ &= \frac{3}{16k}l_0^4(\sum_z \overline{\psi}_z\gamma_\mu\psi_z \sum_{z'}\overline{\psi}_{z'}\gamma^\mu\psi_{z'} + \sum_z \overline{\psi}_z\gamma^5\gamma_\mu\psi_z \sum_{z'}\overline{\psi}_{z'}\gamma^5\gamma^\mu\psi_{z'})g_{\alpha\beta} && \text{(4.12b)}\\ &= \frac{3}{8k}l_0^4 \sum_{z\neq z'}\left(\overline{\psi}_z\gamma_\mu\psi_z\,\overline{\psi}_{z'}\gamma^\mu\psi_{z'} + \overline{\psi}_z\gamma^5\gamma_\mu\psi_z\,\overline{\psi}_{z'}\gamma^5\gamma^\mu\psi_{z'}\right)g_{\alpha\beta}\ , && \text{(4.12c)} \end{aligned}$$

and also contains the energy-momentum tensors of the individual spinor fields ψ_z. The result is

$$\begin{aligned} \frac{1}{k}G^*_{\alpha\beta} = \sum_z \frac{i\hbar c}{4}[&\overline{\psi}_z\gamma_a(\nabla^*_\beta + \varepsilon(z)S_\beta)\psi_z - (\nabla^*_\beta - \varepsilon(z)S_\beta)\overline{\psi}_z\gamma_\alpha\psi_z + (\alpha\leftrightarrow\beta)] \\ &+ \frac{16}{k}l^2[S_{\alpha\gamma}S_\beta{}^\gamma - \frac{1}{4}g_{\alpha\beta}S_{\mu\nu}S^{\mu\nu}] \\ &+ W_{\alpha\beta}\ . \end{aligned} \tag{4.13}$$

If we compare this equation with the corresponding energy-momentum equation (1.7) of the Einstein–Cartan theory, then, besides the additional contributions from vector currents in (4.12), also the doubled factor 3/8 instead of 3/16 occures. This is due to the summation of the various contact interaction terms, where each interaction between two distinct Dirac fields was counted twice, when the basic expression (4.12a) is reexpressed through the individual currents via (4.7) as in (4.12b).

[3]We leave out the detailed computations, since they are rather tedious.

As has been already noted in [Hor 95], the vanishing of the self-interacting terms in (4.12) and also in the Dirac equation (4.11) are formally due to the identity (2.38) and have their origin in our special choice of $\mathcal{L}_m$ in (2.15), where the adjoint covariant derivative of $\overline{\psi}$ is missing. Usually, the matter Lagrangian is required to be real, necessitating the inclusion of both derivatives of ψ and $\overline{\psi}$, cf. [Heh 71]. Since in (2.15) the Lagrangians $\mathcal{L}_G$ and $\mathcal{L}_Y$ were already complex, there was no need to make $\mathcal{L}_m$ alone real valued by including the adjoint spinor derivative. Stated differently, if the Lagrangian of the Einstein–Cartan theory, which normally contains both dervivatives of ψ and $\overline{\psi}$ (see for example [Heh 71]), is changed by omitting the adjoint covariant derivative of $\overline{\psi}$, then the self-interaction terms in (1.7) and in (1.8) will change and become the same as in our theory. Thus, for the single-particle case, these interaction terms will vanish, and we must also consider in the Einstein–Cartan theory a many-particle theory to discover the spin-spin contact interaction.

4.2 Apparent universality of the contact interaction

4.2.1 Einstein–Cartan theory

Kerlick [Ker 75] and Rumpf [Rum 79] concluded that the spin-spin contact interaction of the Einstein–Cartan theory is *universal*, that is, it does not depend on the matter type (whether particles or anti-particles) considered. It is attractive for Dirac fields with opposite spins and repulsive for aligned spins [Ker 75].

To see how these authors argued in this context we briefly discuss the work of Rumpf [Rum 79], p. 649, using our notations. Consider the Dirac equation (1.8) in a special Riemann–Cartan spacetime with flat metric $g_{\alpha\beta} = \eta_{\alpha\beta}$ like in Minkowski spacetime, but with non-vanishing torsion, thus,

$$i\hbar c\gamma^\mu\partial_\mu\psi - mc^2\psi + \frac{3}{8}l_0^2\hbar c j^{5\,\delta}\gamma^5\gamma_\delta\psi = 0\ . \tag{4.14}$$

Here the axial current $\overline{\psi}\gamma^5\gamma^\delta\psi$ has been replaced by a *background* field $j^{5\,\delta}$, so that the spinor ψ loses its cubic self-interaction. This replacement means that the totally antisymmetric torsion field in (1.6) is solely caused by this background field. We may imagine that this axial current is due to a constant classical background Dirac field at rest, which in addition is polarized in the z direction,

$$\psi_{bg} := \sqrt{n}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} e^{-imc^2/\hbar\cdot t}\ , \tag{4.15}$$

where n is the constant particle density. Noting that

$$H := i\hbar c\partial_0 = i\hbar\partial_t \tag{4.16}$$

is the Hamiltonian of the Dirac field ψ in (4.14), we can first compute the axial current $j^{5\,\delta}$ from ψ_{bg} and then reexpress (4.14) as follows (see (B.3a))

$$H\psi = -i\hbar c\gamma^0\vec{\gamma}\cdot\nabla\psi + mc^2\gamma^0\psi + \frac{3}{8}l_0^2\hbar cn\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}\psi\ . \tag{4.17}$$

Here the symbol ∇ denotes the ordinary gradient vector, and $\vec{\gamma}$ stands for $\vec{\gamma} = (\gamma^1, \gamma^2, \gamma^3)$. We can solve this eigenvalue equation by the ansatz of a free wave polarized in the positive (negative) z direction (N is a normalization constant)

$$\psi_\uparrow := N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\exp(-\frac{i}{\hbar}p_\mu x^\mu) \qquad \text{and} \qquad \psi_\downarrow := N\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\exp(-\frac{i}{\hbar}p_\mu x^\mu)\ . \tag{4.18}$$

The energy eigenvalues of these two solutions read

$$E_{\uparrow\uparrow} = mc^2 + \frac{3}{8}l_0^2\hbar cn\ ; \tag{4.19a}$$

$$E_{\uparrow\downarrow} = mc^2 - \frac{3}{8}l_0^2\hbar cn\ , \tag{4.19b}$$

where the arrow-subscipts at E denote the spin directions of the background field and that of the test particle. It may be easily checked that this result remains true if we consider Dirac anti-particles rather than particles as test fields [Ker 75]. Since the energy level is raised (lowered) if the spins are parallel (anti-parallel) we may conclude that the spin-spin contact interaction is repulsive for aligned spins and attractive for opposite spins. Since this feature does not depend whether one considers ordinary matter or anti-matter one may speak of a *universal* spin-spin contact interaction.

4.2.2 The new spin-spin contact interaction

The situation encountered above changes if we consider the new spin-spin contact interaction in (4.12). Instead of (4.14), we now have

$$i\hbar c\gamma^\mu\partial_\mu\psi - mc^2\psi + \frac{3}{8}l_0^2\hbar c(j^\delta + j^{5\,\delta}\gamma^5)\gamma_\delta\psi = 0\ . \tag{4.20}$$

Proceeding in exactly the same manner as above, we obtain for the energies of the test particles (see (B.5))

$$E_{\uparrow\uparrow} = mc^2 \; ; \tag{4.21a}$$

$$E_{\uparrow\downarrow} = mc^2 - 2 \cdot \frac{3}{8} l_0^2 \hbar c n \; . \tag{4.21b}$$

Contrary to the Einstein–Cartan theory discussed above, there is no observable force between aligned spins, whereas the attractive force between opposite spins is now twice as strong as before. Furthermore, now the energy shifts of a test field describing anti-matter in a background torsion are not equal to (4.21), but are given by (see (B.6) and (B.7))

$$E'_{\uparrow\uparrow} = mc^2 + 2 \cdot \frac{3}{8} l_0^2 \hbar c n \; ; \tag{4.22a}$$

$$E'_{\uparrow\downarrow} = mc^2 \; . \tag{4.22b}$$

We see here that opposite spins do not feel any force acting between them. However, the repulsive force between aligned spins turns out to be stronger than in (4.21). From (4.21) and(4.22) it follows that the universality of the contact interaction is lost now: The energy shifts due to the contact interaction between an ordinary background matter field and a test particle describing ordinary matter differs from the case where the test particle describes an anti-matter field.

4.3 Quantizing the contact interaction

In the last section we have obtained the energy shifts of a Dirac field caused by contact interaction terms. This was simply done by finding the energy eigenvalues of the modified Dirac equations. One drawback of this procedure is that the Dirac fields are not second-quantized, so that their Fermi-Dirac statistics are completely disregarded. This is particularly unsatisfactory, since *"the only source of torsion is intrinsic fundamental-particle spin. ... Thus, torsion is fundamentally a microscopic, quantum mechanically related phenomenon"* [Sto 85].

In this section we therefore quantize the contact interaction term and investigate, how the resulting interaction Hamiltonian acts on various quantum two-particle states. In this way, we shall obtain more detailed informations about the shifts of energy levels of Dirac particles. For example, the contact interaction will turn out to be *non-universal* even in the case of Einstein–Cartan theory.

The theoretical method applied for this study is simply the first Born approximation. Thus, we only consider first-order reactions caused by the contact interaction Hamiltonian. It is well-known that four-fermion contact interactions as considered here, which are of the similar structure as the phenomenological Fermi

interaction of the weak forces, lead to non-renormalizable theories. We will argue below why it is yet sufficient to study the contact interaction only in the first Born approximation.

4.3.1 Interaction Hamiltonian

We begin with determining the effective Lagrangian density of the many-particle system considered in the first section. For the sake of simplicity, we take a two-particle system consisting of two arbitrary charged Dirac spinors $\psi_1 = \psi$ and $\psi_2 = \chi$, having masses $m_1 = m$ and $m_2 = n$ and charges ε_1 and ε_2, respectively. Now, if we insert the field equation (2.27) for the connection together with (4.7) into the basic Lagrangian density (4.4), this Lagrangian density may be reexpressed (see eq. (B.14) to (B.17)) as

$$\begin{aligned}\mathcal{L} = g\, i\hbar c[\overline{\psi}\gamma^\mu(\nabla^*_\mu + \varepsilon_1 S_\mu)\psi + \frac{imc}{\hbar}\overline{\psi}\psi] + g\, i\hbar c[\overline{\chi}\gamma^\mu(\nabla^*_\mu + \varepsilon_2 S_\mu)\chi + \frac{inc}{\hbar}\overline{\chi}\chi] \\ - \frac{g}{2k}R^* + g\frac{l^2}{4k}S_{\mu\nu}S^{\mu\nu} + g\frac{3}{8}\frac{l_0^4}{k}[\overline{\psi}\gamma_\mu\psi\,\overline{\chi}\gamma^\mu\chi + \overline{\psi}\gamma^5\gamma_\mu\psi\,\overline{\chi}\gamma^5\gamma^\mu\chi]\ .\end{aligned} \tag{4.23}$$

The last term is the *spin-spin interaction Lagrangian* denoted henceforth by

$$\mathcal{L}_I = g\frac{3}{8}\frac{l_0^4}{k}[\overline{\psi}\gamma_\mu\psi\,\overline{\chi}\gamma^\mu\chi + \overline{\psi}\gamma^5\gamma_\mu\psi\,\overline{\chi}\gamma^5\gamma^\mu\chi]\ . \tag{4.24}$$

Since we want to investigate only this contact interaction, we neglect the effects of gravity and electromagnetism. Thus, we set the charges to zero and employ from now on flat Minkowski spacetime with constant Minkowski metric

$$g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)\ . \tag{4.25}$$

Thus, now the density factor g in (4.24) equals 1 and therefore may be omitted in (4.23).

To obtain the interaction Hamiltonian H_I to be quantized in the following, we must subject (4.24) to the well-known Legendre transformation

$$H = \int d^3x[\sum_a \pi_a \partial_t \phi_a - \mathcal{L}]\ , \tag{4.26}$$

where $\mathcal{L}$ is an arbitrary Lagrangian density, depending on general fields, here denoted by ϕ_a. The fields π_a denote the conjugate fields,

$$\pi_a = \frac{\partial \mathcal{L}}{\partial[\partial_t \phi_a]}\ , \tag{4.27}$$

see for example [Itz 80]. Since in $\mathcal{L}_I$ in (4.24) there are no derivative terms present, we obtain the corresponding interaction Hamiltonian simply by the change of sign,

$$H_I = -\int d^3\!x \mathcal{L}_I = -\frac{3}{8}\frac{l_0^4}{k}\int_V d^3\!x[\overline{\psi}\gamma_\mu\psi\overline{\chi}\gamma^\mu\chi + \overline{\psi}\gamma^5\gamma_\mu\psi\overline{\chi}\gamma^5\gamma^\mu\chi] \, . \tag{4.28}$$

Here the space integration is to be performed only in a volume V in order to obtain finite results later on.

4.3.2 Quantization procedure

Notation

We quantize the Dirac fields ψ and χ using the usual canonical quantization procedure [Itz 80]. Our notation is as follows: The *operators* ψ and χ are expanded in terms of c-number plane wave solutions u, v, s, and w of the ordinary Dirac equations in the Minkowski spacetime and operator-valued amplitudes a, $a^\dagger$, b, $b^\dagger$, c, $c^\dagger$, d, and $d^\dagger$,

$$\psi = \int \frac{d^3\!p}{(2\pi)^3}\frac{m}{p^0}[b^\dagger_{rp}v_r(p)e^{ipx} + a_{rp}u_r(p)e^{-ipx}] \, , \tag{4.29a}$$

$$\chi = \int \frac{d^3\!p}{(2\pi)^3}\frac{n}{p^0}[d^\dagger_{rp}w_r(p)e^{ipx} + c_{rp}s_r(p)e^{-ipx}] \, . \tag{4.29b}$$

We have suppressed the summation over the spin r, which takes the value $r = +\frac{1}{2}$ when the spin is parallel to the positive x^3-direction and the negative value $r = -\frac{1}{2}$ when the spin is anti-parallel. Further, p denotes the 4-momentum with the condition $p^0 = \sqrt{m^2 + \vec{p}^2} > 0$ in (4.29a) and $p^0 = \sqrt{n^2 + \vec{p}^2} > 0$ in (4.29b), respectively. The plane wave solutions u, v, s, and w are explicitly given in (B.19) and were taken from [Nac 90]. The pleasant feature of these plane wave solutions is that the waves describing the anti-matters, v and w, have the right spin directions: For example, $v_{+1/2}(p)$ describes an anti-matter wave solution with its spin in the positive x^3-direction. For our purposes here, we do not need the explicit expressions, but only the normalization conditions. These are given by (see [Nac 90] and (B.19))

$$\bar{u}_r(p)u_{r'}(p) = -\bar{v}_r(p)v_{r'}(p) = V^{-1}\delta_{rr'} \; ; \tag{4.30a}$$

$$\bar{s}_r(p)s_{r'}(p) = -\bar{w}_r(p)w_{r'}(p) = V^{-1}\delta_{rr'} \, , \tag{4.30b}$$

and

$$\bar{u}_r(p)\gamma^\mu u_{r'}(p) = \bar{v}_r(p)\gamma^\mu v_{r'}(p) = V^{-1}\frac{p^\mu}{m}\delta_{rr'} \; ; \tag{4.31a}$$

$$\bar{s}_r(p)\gamma^\mu s_{r'}(p) = \bar{w}_r(p)\gamma^\mu w_{r'}(p) = V^{-1}\frac{p^\mu}{n}\delta_{rr'} \, . \tag{4.31b}$$

(4.31c)

In calculating the energy shifts below, the following *axial vector* expression built from an arbitrary 4-momentum p^μ with intrinsic rest mass $M := \sqrt{p^\mu p_\mu}$ becomes useful:

$$p^{5\mu} := \begin{pmatrix} p^3 \\ p^3 p^1/(p^0+M) \\ p^3 p^2/(p^0+M) \\ p^3 p^3/(p^0+M)+M \end{pmatrix} . \tag{4.32}$$

We can then employ this axial vector to express the axial currents of the plain wave solutions, see eqs. (B.20) to (B.23)

$$\bar{u}_r(p)\gamma^5\gamma^\mu u_r(p) \;=\; -\bar{v}_r(p)\gamma^5\gamma^\mu v_r(p) \;=\; \mp V^{-1}\frac{1}{m}p^{5\mu} \quad \cdots r=\pm\frac{1}{2}\,; \tag{4.33a}$$

$$\bar{s}_r(p)\gamma^5\gamma^\mu s_r(p) \;=\; -\bar{w}_r(p)\gamma^5\gamma^\mu w_r(p) \;=\; \mp V^{-1}\frac{1}{n}p^{5\mu} \quad \cdots r=\pm\frac{1}{2}\,. \tag{4.33b}$$

Note that $p^{5\mu}$ are not equal in the both formulae, since the rest masses of the particles are different in (4.32). Also, if we insert the explicit formula (4.32) into (4.33), we see that the p^3 component of the 4-momentum p^μ is not treated in the same way as the other two space-like components p^1 and p^2. The reason is simply that we have chosen the plane wave solutions to be polarized entirely in the x^3 direction and thus distinguished this space direction.

The operator-valued amplitudes a, $a^\dagger$, b, $b^\dagger$, c, $c^\dagger$, d, and $d^\dagger$ satisfy the anti-commutation relations [Nac 90]

$$\{a_{rp}, a^\dagger_{r'p'}\} \;=\; \{b_{rp}, b^\dagger_{r'p'}\} \;=\; (2\pi)^3\frac{p^0}{m}\delta^3(\vec{p}-\vec{p'})\delta_{rr'}\,; \tag{4.34a}$$

$$\{c_{rp}, c^\dagger_{r'p'}\} \;=\; \{d_{rp}, d^\dagger_{r'p'}\} \;=\; (2\pi)^3\frac{p^0}{n}\delta^3(\vec{p}-\vec{p'})\delta_{rr'}\,, \tag{4.34b}$$

where all other anti-commutations vanish. The interpretations of these operators are as usual: For example, $a^\dagger_{rp}$ generates a particle having 4-momentum p and x^3-component of spin r, whereas $b^\dagger_{rp}$ generates the corresponding anti-particle state with the spin also in the x^3-direction.

Quantization

We quantize the interaction Hamiltonian (4.28) according to the canonical quantization procedure. Thus we merely replace the fields ψ and χ and their adjoint fields by the corresponding operator expressions (4.29a) and (4.29b), respectively. However, to yield finite results, we must *normal-order* the operator expressions. Denoting this normal-ordering by : we obtain the following interaction Hamiltonian operator:

$$\widehat{H}_I = -\frac{3}{8}\frac{l_0^4}{k}\int_V d^3\!x\,[\,:\overline{\psi}\gamma_\mu\psi\,\overline{\chi}\gamma^\mu\chi: + :\overline{\psi}\gamma^5\gamma_\mu\psi\,\overline{\chi}\gamma^5\gamma^\mu\chi:\,]\,. \tag{4.35}$$

We remark that this operator expression does not vanish even if χ is replaced by ψ and $\overline{\chi}$ by $\overline{\psi}$, that is, if the contact interaction is considered between the same kind of particles. This feature differs of course from the classical expression H_I in (4.28), which vanishes for identical Dirac fields due to the Pauli relation (2.38). Note that, if we had written the interaction Hamiltonian operator as follows,

$$\widehat{H}_I = -\frac{3}{8}\frac{l_0^4}{k}\int_V d^3x [: \overline{\psi}\gamma_\mu\psi :: \overline{\chi}\gamma^\mu\chi : + : \overline{\psi}\gamma^5\gamma_\mu\psi :: \overline{\chi}\gamma^5\gamma^\mu\chi :] \,, \tag{4.36}$$

we would have obtained infinite results, because the field operators ψ and $\overline{\psi}$ on the one hand and χ and $\overline{\chi}$ on the other hand are taken at the same spacetime point, as can be verified by an explicit computation.

Note also that the corresponding interaction Hamiltonian operator $\widehat{H}_{ECT}$ for the spin-spin contact interaction in the Einstein–Cartan theory reads

$$\widehat{H}_{ECT} = -\frac{3}{8}\frac{l_0^4}{k}\int_V d^3x [: \overline{\psi}\gamma^5\gamma_\mu\psi\overline{\chi}\gamma^5\gamma^\mu\chi :] \,. \tag{4.37}$$

Whereas the vanishing of the vector current contribution in comparison with (4.35) is clear from the classical interaction expression in (1.7), the appearance of the same factor 3/8 seems strange, if we compare the classical expressions (1.7) with (4.12). However, to study the spin-spin contact interactions of our theory and that of the Einstein–Cartan theory on the equal basis, we must consider in the Einstein–Cartan theory also a many-particle theory instead of a single-particle theory as presented in the introducing chapter. If this is done, then the torsion $T_{\alpha\beta\gamma}$ in (1.6) is no longer produced by only one Dirac field ψ but by the sum of many different fields. In the same fashion as explained in connection with (4.13), this leads to the double factor 3/8 in (1.7) instead of 3/16.

4.3.3 Evaluation on two-particle states

To investigate the energy shifts of the two-particle system (4.23) due to the contact interaction we evaluate the expectation values of the interaction Hamiltonian $\widehat{H}_I$ between the following two-particle states:

$$|1\rangle := c^\dagger_{rq}a^\dagger_{r'q'}|0\rangle \; ; \tag{4.38a}$$
$$|2\rangle := d^\dagger_{rq}b^\dagger_{r'q'}|0\rangle \; ; \tag{4.38b}$$
$$|3\rangle := c^\dagger_{rq}b^\dagger_{r'q'}|0\rangle \; ; \tag{4.38c}$$
$$|1o\rangle := a^\dagger_{rq}a^\dagger_{r'q'}|0\rangle \; ; \tag{4.38d}$$
$$|2o\rangle := b^\dagger_{rq}b^\dagger_{r'q'}|0\rangle \; ; \tag{4.38e}$$
$$|3o\rangle := b^\dagger_{rq}a^\dagger_{r'q'}|0\rangle \,. \tag{4.38f}$$

The first state $|1\rangle$ consists of two particles of different kind, the second state $|2\rangle$ is built from two anti-particles, and the third state $|3\rangle$ contains one particle and one antiparticle. The last three states $|1o\rangle$, $|2o\rangle$ and $|3o\rangle$ describe corresonding two-particle states consisting of two identical particles (but of course in different states).

The calculations of the expectation values of $\widehat{H}_I$ (4.35) are standard, see B.4. We introduce the symbol

$$\sigma(r,r') := \begin{cases} +1 \text{ if } r = r' \\ -1 \text{ if } r \neq r' \end{cases} \tag{4.39}$$

and obtain

$$\begin{aligned}
\langle 1|\widehat{H}_I|1\rangle &= -\frac{3}{8}\frac{l_0^4}{kVmn}[\ \ q^\mu q'_\mu + \sigma(r,r')q^{5\mu}q'^5_\mu]\ ; &(4.40\text{a})\\
\langle 2|\widehat{H}_I|2\rangle &= -\frac{3}{8}\frac{l_0^4}{kVmn}[\ \ q^\mu q'_\mu + \sigma(r,r')q^{5\mu}q'^5_\mu]\ ; &(4.40\text{b})\\
\langle 3|\widehat{H}_I|3\rangle &= -\frac{3}{8}\frac{l_0^4}{kVmn}[-q^\mu q'_\mu + \sigma(r,r')q^{5\mu}q'^5_\mu]\ ; &(4.40\text{c})\\
\langle 1o|\widehat{H}_I|1o\rangle &= -\frac{3}{2}\frac{l_0^4}{kVm^2}[\ \ q^\mu q'_\mu + \sigma(r,r')q^{5\mu}q'^5_\mu]\ ; &(4.40\text{d})\\
\langle 2o|\widehat{H}_I|2o\rangle &= -\frac{3}{2}\frac{l_0^4}{kVm^2}[\ \ q^\mu q'_\mu + \sigma(r,r')q^{5\mu}q'^5_\mu]\ ; &(4.40\text{e})\\
\langle 3o|\widehat{H}_I|3o\rangle &= -\frac{3}{2}\frac{l_0^4}{kVm^2}[-q^\mu q'_\mu + \sigma(r,r')q^{5\mu}q'^5_\mu]\ . &(4.40\text{f})
\end{aligned}$$

In the first three expectation values, which are taken for two different kinds of particles, the first term $p^\mu p'_\mu$ exactly corresponds to the vector-vector term in $\widehat{H}_I$ (4.35), whereas the second summand $\sigma(r,r')q^{5\mu}q'^5_\mu$ is exactly the axial-axial term, cf. (4.33). On the other hand, such a simple decomposition does not apply for the last three expectation values based on two identical particles: In order to obtain these simple expressions (4.40d) to (4.40f), one has to use the Fierz transformation rule (see B.4) to order the entanglement of the various plain waves, which has its origin in the exchange degeneracy of identical fermions. Also, due to the greater statistical freedom of an identical particle system, the last three expectation values are 4 times the first three expressions.

The expectation values for the Hamiltonian $\widehat{H}_{ECT}$ (4.37) of the Einstein–Cartan theory read (cf. B.4)

$$\begin{aligned}
\langle 1|\widehat{H}_{ECT}|1\rangle &= -\frac{3}{8}\frac{l_0^4}{kVmn}[\sigma(r,r')q^{5\mu}q'^5_\mu]\ ; &(4.41\text{a})\\
\langle 2|\widehat{H}_{ECT}|2\rangle &= -\frac{3}{8}\frac{l_0^4}{kVmn}[\sigma(r,r')q^{5\mu}q'^5_\mu]\ ; &(4.41\text{b})
\end{aligned}$$

$$\langle 3|\widehat{H}_{ECT}|3\rangle = -\frac{3}{8}\frac{l_0^4}{kVmn}[\sigma(r,r')q^{5\mu}q'^5_\mu] ; \tag{4.41c}$$
$$\langle 1o|\widehat{H}_{ECT}|1o\rangle = -\frac{3}{8}\frac{l_0^4}{kVm^2}[2m^2 + q^\mu q'_\mu + 3\sigma(r,r')q^{5\mu}q'^5_\mu] ; \tag{4.41d}$$
$$\langle 2o|\widehat{H}_{ECT}|2o\rangle = -\frac{3}{8}\frac{l_0^4}{kVm^2}[2m^2 + q^\mu q'_\mu + 3\sigma(r,r')q^{5\mu}q'^5_\mu] ; \tag{4.41e}$$
$$\langle 3o|\widehat{H}_{ECT}|3o\rangle = -\frac{3}{8}\frac{l_0^4}{kVm^2}[2m^2 - q^\mu q'_\mu + 3\sigma(r,r')q^{5\mu}q'^5_\mu] . \tag{4.41f}$$

The first three expressions (4.41a) to (4.41c) can be obtained simply by neglecting the vector-vector interacton parts in the corresponding results (4.40a) to (4.40c). This can be directly understood in terms of the underlying expressions for the interaction Hamiltonian $\widehat{H}_I$ and $\widehat{H}_{ECT}$, which differ only in the vector-vector interaction term. On the other hand, such a simple understanding can not be given for the last three expressions in (4.41).

4.3.4 Discussion

Let us first discuss the contact interaction between different kinds of particles. Whereas (4.40a) to (4.40b) are obviously non-universal, the corresponding expectation values of the Einstein–Cartan theory (4.41a) to (4.41c) are universal, that is, they do not depend on whether one considers particles or anti-particles. Furthermore, if both interacting particles are at rest, then $q^{5\mu}q'^5_\mu = -mn$, as can be easily verified with the help of (4.32). Thus in this special case, the interaction energy increases (decreases) for aligned (opposite) spins in accordance with the result of the consideration in (4.19).

But when two identical particles interact, then we see from (4.41d) to (4.41f), that the contact interaction fails to be universal also in the Einstein–Cartan case. Contrary to the new spin-spin contact interaction in (4.40d) to (4.40f), where merely a symmetry factor 4 is needed to adjust the formulae to the case of identical particles, the interaction energy in the Einstein–Cartan theory gains some miscellaneous contributions due to the Fierz transformation (see (B.28)). For example, let us consider non-relativistic identical particles both having no momentum in the x^3-direction, that is, $|\vec{q}|, |\vec{q'}| \ll m$ and $q^3 = q'^3 = 0$. Then, from (B.30), we obtain for the Einstein–Cartan case

$$\langle 1o|\widehat{H}_{ECT}|1o\rangle = \frac{3}{8}\frac{l_0^4}{kV}[3(\sigma(r,r') - 1) - \frac{(\vec{q}-\vec{q'})^2}{2m^2}] , \tag{4.42}$$

and, for the new contact interaction, a quite similar result:

$$\langle 1o|\widehat{H}_I|1o\rangle = \frac{3}{8}\frac{l_0^4}{kV}[4(\sigma(r,r') - 1) - 4\frac{(\vec{q}-\vec{q'})^2}{2m^2}] . \tag{4.43}$$

We observe, that in both cases the interaction energy is negative. Thus, it is possible that the contact interactions among identical particles with aligned spins could be attractive in contrast to the statement made by Kerlick, cf. (4.21).

We may say that the spin-spin contact interaction of the Einstein–Cartan theory is in general *not universal* and it is not always true that aligned spins repell.

Note that the new spin-spin contact interaction does not allow for self-interactions among spinors already on the classical level in contrast to the ordinary contact interaction of the Einstein–Cartan theory.

4.3.5 Justification of the first Born approximation

The Born approximation of the contact interaction can also be found in the work by Kanno [Kan 88]. But contrary to our approach, he computed the expectation value of the contact interaction energy for many-particle states with high matter density. Thereby he assumed that these states can be approximated by summing up the free wave states of each particles. He concluded that there occurs a matter–anti-matter segregation due to the contact interaction. In my opinion, his approach is not correct since at high densities, a quantum-mechanical many-particle system with torsion can not be approximated by a sum of plane wave states: At high matter density, the torsion (1.6) becomes non-negligible and changes the Dirac equation (1.8) significantly. Therefore, plane wave solutions of the ordinary Dirac equation (without the cubic interaction term) do not approximate solutions of the Dirac equation with torsion. Thus, it makes no sense to take an expectation value of the interaction Hamiltonian between free wave states, since no such states exist at high density.

On the other hand, we have studied the contact interaction between two particles, so that the matter density is negligible and the plane wave solutions really approximate the solutions of the Dirac equation (cf. (4.10)) very well.

Let us now justify why it is legitimate to consider only the first Born approximation of the contact interaction. It is well known that the phenomenological Fermi contact interaction describes the weak interaction very well at low momenta. To be more precise, the description of the weak force by the contact interaction is valid for energies up to the critical value $1/\sqrt{G_{Fermi}} \approx 300$GeV, see for example [Itz 80]. If this value is exceeded, the phenomenological contact interaction violates the unitarity. Now, since the torsion-induced spin-spin contact interaction has a coupling constant, which is of the order of the squared Planck length[4] it is legitimate to consider the first Born approximation for energies up to the Planck energy $2.4 \cdot 10^{18}$GeV. But at this enormously high energy, or, equivalently, at the Planck scale, we would need a quantum theory of gravity to describe the physics properly. If we restrict

[4] Note that in (4.25) $l_0^4/k = l_0^2 \cdot \hbar c$.

ourselves to energies below the Planck scale, then the first Born approximation of the contact interaction is physically sensible.

Chapter 5

Summary and Outlook

5.1 Summary

In the preceeding chapters I have reexamined and improved the unified field theory of gravity and electromagnetism developed in my diploma thesis [Hor 94]. Furthermore, the special spin-spin contact interaction predicted by this theory was investigated in detail.

Although the theory presented here was motivated by earlier works on unified field theories [Bor 76a, Mof 77, Kun 79, McK 79, Fer 82, Jak 85], in which the torsion trace T_μ of a real linear connection was identified with the electromagnetic vector potential A_μ, the new theory comes to completely different conclusions: In this new theory, the linear connection resulting from the field equations is complex-valued and it is not compatible with the metric, where this failure of compatibility is caused by a vector part S_μ of the connection (the so-called non-metricity vector). According to the geometrical background of the new theory, this vector S_μ can be unambiguously detached from the tangent frame bundle of the spacetime manifold and then identified with the electromagnetic vector potential on a trivial U(1) bundle. Contrary to this truly geometric identification, the relation between the torsion trace and the vector potential on the tangent frame bundle can be obtained only if a special U(1) gauge is chosen and held fixed on the U(1) bundle. For this reason, the long-standing relation between the torsion trace and the electromagnetic potential is merely a formal consequence of the geometrical background underlying the new theory. Furthermore, due to this geometry the whole complex connection resulting from the field equations can be decomposed into the vector potential S_μ and a Lorentzian connection compatible with the metric, this being done by means of pull-back techniques. If we consider the torsion trace of this Lorentzian connection part only, it is not related to electromagnetism even formally. Thus, in the end, the torsion trace is not related to electromagnetic phenomena at all. However, it is important to note that this conclusion can only be drawn with the help of an

investigation of the special geometrical background of the new theory.

This geometrical background has been explained in chapter 3 in every detail, thereby clarifying several difficult properties, which were not mentioned in the diploma thesis [Hor 94]: First, the notion of a complex spin geometry and its relation to the usual spin geometry has been explained rigorously. Secondly, the pull-back procedure, by which an unique spinor derivative can be obtained from any complex linear connection, has been improved. Thereby it was shown why the U(1) principal bundle accounting for the electromagnetic phase transformation is necessarily trivial in the geometrical framework of our theory. Also, the roles played by different "intermediate" bundles in this pull-back procedure has been clarified in detail. Thirdly, the decomposition principle of the linear connection, by means of which it is possible to obtain a meaningful theory of electromagnetism, has been elaborated. Forthly, the properties of the U(1) gauge transformation in the geometrical framework has been investigated. From this gauge structure, we were able to see why it is necessary to detach the U(1) vector potential from the tangent frame bundle and to pull it back onto a trivial U(1) bundle: If, instead, the U(1) gauge transformation is considered on the basic tangent frame bundle of the spacetime manifold, every covariant vector field gains a negative elementary charge, which is clearly unphysical. Also, the same reasons show why it is impossible to introduce a formal U(1) gauge transformation, the so-called λ–transformation, for the torsion trace.

Besides these electromagnetic and geometrical aspects, the new theory also incorporates a spin-spin contact interaction between spinning particles. This property is shared by none of the unified field theories proposed before and is one of the salient features of the new theory. The contact interaction is also a characteristic feature of the Einstein–Cartan theory, and has its origin in the spin-torsion coupling, by which the spacetime geometry can not only respond to mass-energy via the curvature, but also to spin via torsion. These properties of the spacetime together with the geometric interpretation of electromagnetism of our new gravitational field theory lead to the conclusion that the spacetime geometry is able to interact with three basic features of elementary particles: Mass-energy, spin, and electromagnetic charge.

Contrary to the ordinary axial current contact interaction of the Einstein–Cartan theory, the new contact interaction has contributions from both the axial and vector currents of Dirac spinors. This has the effect that now there are no self-interactions among Dirac fields as was the case in the Einstein–Cartan theory. This feature respects the quantum nature (Fermi–Dirac statistics) of elementary fermions already on the classical, i.e. not second-quantized, level, and makes the new contact interaction more favourable than the ordinary one. By regarding the energy eigenvalues of test fields in an background torsion field, the new spin-spin interaction turns out to be non-universal in contrast to the ordinary one: Now the interacting force between particle fields is different from the corresponding force be-

tween a particle and an anti-particle field. This difference persists if the contact interaction is quantized.

The contact interaction has been investigated further on the quantum level by means of the first Born approximation, similar to the phenomenological Fermi contact interaction of the weak forces. It turned out that both the new and the ordinary contact interactions, upon quantization, are non-universal in the case of identical particles interacting with each other. And in this case, if the particle momenta are small, both contact interactions have similar structure and are attractive regardless of the spin directions of the interacting particles. This result is in sharp contrast to the common opinion [Ker 75], that the contact interaction is attractive only if the spins are opposed, but repulsive if they are in alignment.

5.2 Future research

5.2.1 Weak interaction

Since the unified field theory considered in this work enables the spacetime geometry to interact with three fundamental properties of elementary particles, namely mass, spin, and charge, it is natural to ask whether it is possible to incorporate the weak forces into the geometrical framework provided by this theory.

If we stay on the non-quantum level, this can be hardly achieved by the present theory itself, since the theory does not contain charged vector boson fields as required for the Weinberg–Salam theory. Thus, it seems necessary to further enlarge the spacetime geometry, using, for example, an arbitrary covariant spinor derivative instead of a spinor derivative built from a complex linear connection. This speculation is confirmed by a survey of unified theories of gravity and electroweak interaction based upon the "geometry of the tangent bundle" instead of a spin structure: Without exception these theories [Bor 76b, Nov 85, Bat 84, Yil 89, Bat 90, Nov 92] are not acceptable as realistic physical theories.

The idea of using an enlarged spin structure for the unification of gravity and electroweak forces is not new and has already been considered in many works, see e.g. [Nov 73, Tro 87, Chi 87, Chi 89]. But in all of these works, there is one significant problem which could not be solved rigorously: In the Weinberg–Salam theory, the charged W-boson couples to electron and neutrino via the interaction term (cf. [Ren 90])

$$\sim \overline{\psi}_\nu \gamma^\mu (1-\gamma^5) \psi_e W^+_\mu \ .$$

To obtain such an interaction between different spinor fields and a connection part using the concept of an enlarged spinor derivative, it is necessary to introduce a SU(2) theory of the pair of spinors (ψ_ν, ψ_e) explicitly or in a different, more indirect manner. But if this procedure is followed, the "unified field theories" of gravity and

the weak forces are by no means superior to the standard Weinberg–Salam theory itself, because such a "unified theory" can also be obtained much more easily by embedding the standard model into the pseudo-Riemannian geometry of general relativity and adding the Einstein–Hilbert Lagrangian $-\frac{g}{2k}R^*$ to the Weinberg–Salam Lagrangian.

Thus, the concept of an enlarged spin geometry alone would not lead to a satisfactory unification of gravity and weak interaction. In my opinion, it is necessary to consider a quantum field theoretic approach together with the enlarged spin geometry rather than a classical field theory alone. First hints in this direction are provided by the dynamical electroweak symmetry breaking (for a recent review see [Kin 95]), where the electroweak symmetry is broken by a vacuum condensate of fermions. This fermion condensate has its origin in a four-fermion contact interaction (cf. [Lal 92]) like in the Nambu–Jona-Lasinio model [Nam 61]. Since the new spin-spin contact interaction of our theory is very similar to the contact interactions considered in the theories on dynamical symmetry breaking, it seems possible that the torsion of the spacetime geometry is related to electroweak symmetry breaking.

5.2.2 Contact interaction

So far, the effects of the spin-spin contact interaction on cosmology have been examined mainly on the classical, that is, non-quantized, level, see for example the references [Kop 72, Tra 73, Heh 74, Ker 75, Kuc 76, Kuc 78, Nur 83]. Also, the quantum approach of [Kan 88] does not seem to be consistent, as we have argued in 4.3.5.

To study the effects of the spin-spin contact interactions in the early epoch of the universe, where the matter density was enormously high, we must evaluate the thermodynamical average of the interaction energies due to the contact interactions. This is not as straightforward as in the ordinary case, since the contributions from the non-flat metric and the spacetime curvature can not be neglected.

Another interesting point is the following: The spin-spin contact interaction modifies the energy-momentum equation of ordinary general relativity by the tensor $W_{\alpha\beta}$, which has a form similar to the contribution $\Lambda g_{\alpha\beta}$ of the cosmological constant Λ in the Einstein's field equation, cf. (4.12). If this similarity is taken seriously, then we would obtain a cosmological constant, which is proportional to the current-current interaction terms in $W_{\alpha\beta}$. This would imply a time-dependent cosmological constant, whose value would have been very high in the early epoch of the universe, where the matter density has been very high, and whose actual value for the present universe is nearly zero. Such a time-dependent cosmological constant is supported by string theoretic considerations [Lop 95].

Appendix A

4-Vector Decomposition

Let $\Sigma_{\alpha\beta\gamma}$ be an arbitrary third rank tensor, which might be real or complex valued. Given $\Sigma_{\alpha\beta\gamma}$ we can define four contravariant vectors Q_α, S_β, U_γ, and V_δ and a tensor rest $\Upsilon_{\alpha\beta\gamma}$ in the following way:

$$\begin{aligned}
\Sigma_{\alpha\beta\gamma} \;=:\; & \frac{1}{18}\Big[\ (5\Sigma_{\alpha}{}^{\epsilon}{}_{\epsilon} \;-\Sigma^{\epsilon}{}_{\alpha\epsilon} \;-\Sigma^{\epsilon}{}_{\epsilon\alpha})g_{\beta\gamma} \\
& \qquad +(-\Sigma_{\beta}{}^{\epsilon}{}_{\epsilon}+5\Sigma^{\epsilon}{}_{\beta\epsilon}-\Sigma^{\epsilon}{}_{\epsilon\beta})g_{\alpha\gamma} \\
& \qquad +(-\Sigma_{\gamma}{}^{\epsilon}{}_{\epsilon}-\Sigma^{\epsilon}{}_{\gamma\epsilon}+5\Sigma^{\epsilon}{}_{\epsilon\gamma})g_{\alpha\beta}\Big]+\Sigma_{[\alpha\beta\gamma]}+\Upsilon_{\alpha\beta\gamma} && \text{(A.1)} \\
=:\; & \quad Q_\alpha\, g_{\beta\gamma} \;+\; S_\beta\, g_{\alpha\gamma} \;+\; U_\gamma\, g_{\alpha\beta} \;-\; \frac{1}{12}\eta_{\alpha\beta\gamma\delta}V^\delta \;+\; \Upsilon_{\alpha\beta\gamma}\ , && \text{(A.2)}
\end{aligned}$$

where $\Sigma_{[\alpha\beta\gamma]}$ means the antisymmetrization of its indices, and V^δ is given by $V_\delta = 2\eta_{\alpha\beta\gamma\delta}\cdot\Sigma^{\alpha\beta\gamma}$, $\eta_{\alpha\beta\gamma\delta}$ being the volume element (2.4). Note that this decomposition is possible if and only if a metric $g_{\mu\nu}$ is given. From (A.1) we conclude

$$\begin{aligned}
\Sigma_{\alpha}{}^{\epsilon}{}_{\epsilon} \;=\; & \frac{1}{18}\Big[20\Sigma_{\alpha}{}^{\epsilon}{}_{\epsilon}-4\Sigma^{\epsilon}{}_{\alpha\epsilon}-4\Sigma^{\epsilon}{}_{\epsilon\alpha} \\
& \qquad -\Sigma_{\alpha}{}^{\epsilon}{}_{\epsilon}+5\Sigma^{\epsilon}{}_{\alpha\epsilon}-\Sigma^{\epsilon}{}_{\epsilon\alpha} \\
& \qquad -\Sigma_{\alpha}{}^{\epsilon}{}_{\epsilon}-\Sigma^{\epsilon}{}_{\alpha\epsilon}+5\Sigma^{\epsilon}{}_{\epsilon\alpha}\Big]+\Upsilon_{\alpha}{}^{\epsilon}{}_{\epsilon} \quad\Leftrightarrow\quad \Upsilon_{\alpha}{}^{\epsilon}{}_{\epsilon}=0\ . && \text{(A.3)}
\end{aligned}$$

Similarly, $\Upsilon^{\epsilon}{}_{\beta\epsilon}=\Upsilon^{\epsilon}{}_{\epsilon\gamma}=0$. Furthermore from (A.1) it also follows

$$\Sigma_{[\alpha\beta\gamma]}=\Sigma_{[\alpha\beta\gamma]}+\Upsilon_{[\alpha\beta\gamma]} \quad\Leftrightarrow\quad \Upsilon_{[\alpha\beta\gamma]}=0\ . \tag{A.4}$$

This tensor rest $\Upsilon_{\alpha\beta\gamma}$ has no trace part nor an antisymmetric part. Since the original tensor $\Sigma_{\alpha\beta\gamma}$ has $4^3=64$ degrees of freedom (in the real case) and the four vectors take away only 16 degrees, the tensor rest still has 48 degrees of freedom. An explicit example for such a tensor rest is given by

$$\Upsilon_{\alpha\beta\gamma}=\nabla^*_\alpha B_{\beta\gamma}-\frac{1}{3}(C_\beta g_{\alpha\gamma}-C_\gamma g_{\alpha\beta})\ , \tag{A.5}$$

where $B_{\beta\gamma}$ is an antisymmetric tensor given by the total differential of a vector field B_μ by $B_{\beta\gamma} = \partial_\beta B_\gamma - \partial_\gamma B_\beta$, and C_β is another vector field satisfying an "inhomogeneous Maxwell equation" $C^\mu = \nabla^*_\nu B^{\mu\nu}$. It is easy to verify the conditions (A.3) and (A.4). From this example we may conclude that the tensor rest possibly contains interesting structures. But these are of no relevance to our theory yet since $\Upsilon_{\alpha\beta\gamma}$ does not couple to spinorial matter, see (2.34), but appears only in the gravitational Lagrangian part $\mathcal{L}_G$, see (2.22).

Appendix B

Computations in Chapter 4

B.1 Non-quantized Dirac field

Throuout this chapter we employ the Dirac representation defined by

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\ , \quad \gamma^a = \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}\ , \ a = 1,2,3\ , \quad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\ , \tag{B.1}$$

where the Pauli matrices can be found in (3.37). For the special background Dirac spinor given in (4.15) we immediately obtain the results

$$(j^d) \;=\; (\overline{\psi}_{bg}\gamma^d\psi_{bg}) \;=\; n \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\ ; \tag{B.2a}$$

$$(j^{5d}) \;=\; (\overline{\psi}_{bg}\gamma^5\gamma^d\psi_{bg}) \;=\; n \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}\ . \tag{B.2b}$$

Note that the vector current j^d is time-like, whereas the axial current j^{5d} is a space-like vector. Using these expressions, we calculate

$$\begin{aligned} -\gamma^0 \cdot \frac{3}{8} l_0^2 \hbar c \cdot j^{5d}\gamma^5\gamma_d \;&=\; -\gamma^0 \cdot \frac{3}{8} l_0^2 \hbar c n \cdot (-1)\gamma^5(-\gamma^3) \\ &=\; \frac{3}{8} l_0^2 \hbar c n \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ &=:\; \frac{3}{8} l_0^2 \hbar c n \cdot \tau_{ECT}\ ; \end{aligned} \tag{B.3a}$$

$$
\begin{aligned}
-\gamma^0 \cdot \frac{3}{8} l_0^2 \hbar c (j^d \gamma_d + j^{5d} \gamma^5 \gamma_d) &= \frac{3}{8} l_0^2 \hbar c n \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \\
&= \frac{3}{8} l_0^2 \hbar c n \cdot \tau_I \; . \qquad \text{(B.3b)}
\end{aligned}
$$

The minus sign in front of γ^3 comes from $\gamma_a = \eta_{ab}\gamma^b$. With the result (B.3a) it is straightforward to obtain (4.17) from (4.14).

We do not compute the energy eigenvalues of the Einstein–Cartan theory given in (4.19) but only those of the new contact interaction given in (4.21) and (4.22), since the computations are very similar. For this purpose, we replace in (4.17) the interacting matrix τ_{ECT} by τ_I in (B.3b) and put in the special plane wave spinor $\psi_\uparrow$ (4.18). Since we will only consider test particles at rest, we assume in $\psi_\uparrow$ that the 3-momentum (p_1, p_2, p_3) vanishes. We then obtain

$$
cp^0 \psi_\uparrow = (mc^2 \gamma^0 + \frac{3}{8} l_0^2 \hbar c n \tau_I) \psi_\uparrow \; . \qquad \text{(B.4)}
$$

Denoting the total energy by $E_{\uparrow\uparrow} = cp^0$, this eigenvalue equation is equivalent to

$$
E_{\uparrow\uparrow} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} mc^2 & 0 & 0 & 0 \\ 0 & mc^2 - \frac{3}{4} l_0^2 \hbar c n & 0 & 0 \\ 0 & 0 & -mc^2 & 0 \\ 0 & 0 & 0 & -mc^2 - \frac{3}{4} l_0^2 \hbar c n \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \qquad \text{(B.5)}
$$

Thus we obtain $E_{\uparrow\uparrow} = mc^2$ (4.21a). Similarly, $E_{\uparrow\downarrow} = mc^2 - 2 \cdot \frac{3}{8} l_0^2 \hbar c n$ (4.21b).

To obtain (4.22), we use the anti-matter plane waves given by

$$
\psi'_\uparrow = N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \exp(+\frac{i}{\hbar} p_\mu x^\mu) \qquad \text{and} \qquad \psi'_\downarrow = N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \exp(+\frac{i}{\hbar} p_\mu x^\mu) \; , \qquad \text{(B.6)}
$$

where the up-arrow and down-arrow indicate the spin directions of the plane waves cf. [Itz 80]. Using these plane waves instead of $\psi_\uparrow$ and $\psi_\downarrow$ we immediately obtain instead of (B.4)

$$
- cp^0 \psi'_\uparrow = (-mc^2 \gamma^0 + \frac{3}{8} l_0^2 \hbar c n \tau_I) \psi'_\uparrow \; , \qquad \text{(B.7)}
$$

from which we obtain (4.22a). The interaction energy for opposite spins (4.22b) can be derived in a similar fashion.

B.2 Interaction Hamiltonian

In this section we shall derive the effective Lagrangian (4.23) in order to obtain the interaction Lagrangian $\mathcal{L}_I$ in (4.24). The original Lagrangian is given by (4.4), which contains two spinor fields ψ and χ and their adjoint fields.

First, we introduce the abbreviations

$$j^a = \overline{\psi}\gamma^a\psi\ , \quad k^a = \overline{\chi}\gamma^a\chi\ , \quad j^{5a} = \overline{\psi}\gamma^5\gamma^a\psi\ , \quad k^{5a} = \overline{\chi}\gamma^5\gamma^a\chi\ . \tag{B.8}$$

Next, we write down the connection components obtained by considering its field equation. From (4.11), we obtain

$$\Gamma^a{}_{\mu b} = \{^a{}_{\mu b}\} + (Q^a e_{b\mu} + U_b e^a{}_\mu - \frac{1}{12}\eta^a{}_{\mu bd}V^d) + S_\mu \delta^a{}_b \tag{B.9}$$

with

$$-Q^a = U^a = -\frac{il_0^2}{4}(j^a + k^a)\ , \quad V^d = 3l_0^2(j^{5d} + k^{5d})\ . \tag{B.10}$$

We now insert this result step by step into the three Lagrangian parts $\mathcal{L}_m$, $\mathcal{L}_G$ and $\mathcal{L}_Y$ given by (4.4), see also (2.15a). The matter Lagrangian $\mathcal{L}_m$ consists of the two individual Lagrangians for the spinors ψ and χ, denoted henceforth by $\mathcal{L}_{m\psi}$ and $\mathcal{L}_{m\chi}$, respectively. We have

$$\begin{aligned}
\mathcal{L}_{m\psi} &= g\cdot\hbar c[i\overline{\psi}\gamma^\mu(\partial_\mu - \frac{1}{4}\Gamma_{a\mu b}\sigma^{ba} + \frac{\varepsilon_1}{4}\Gamma^a{}_{\mu a})\psi - \frac{mc}{\hbar}\overline{\psi}\psi] \\
&= g\cdot\hbar c[i\overline{\psi}\gamma^\mu(\nabla^*_\mu + \varepsilon_1 S_\mu)\psi - \frac{mc}{\hbar}\overline{\psi}\psi] \\
&\quad + g\cdot\hbar c i[-\frac{1}{4}(Q_a\eta_{cb} + U_b\eta_{ca} - \frac{1}{12}\epsilon_{acbd}V^d)\overline{\psi}\gamma^c\sigma^{ba}\psi]\ .
\end{aligned} \tag{B.11}$$

To evaluate the last term in (B.11), we use the following algebraic identity among γ-matrices (cf. [Heh 71], see also [Hor 94])

$$\gamma^c\gamma^b\gamma^a = \gamma^c\eta^{ba} + \gamma^a\eta^{bc} - \eta^{ac}\gamma^b + i\gamma^5\gamma_d\epsilon^{cbad}\ . \tag{B.12}$$

Remembering that $\sigma^{ba} = \frac{1}{2}(\gamma^b\gamma^a - \gamma^a\gamma^b)$, we get for the last term in (B.11)

$$\begin{aligned}
&-g\cdot\frac{\hbar c i}{4}[(Q_a\eta_{cb} - Q_b\eta_{ca} - \frac{1}{12}\epsilon_{acbd}V^d)(j^a\eta^{bc} - \eta^{ac}j^b + ij^5{}_d\epsilon^{cbad})] \\
&= -g\cdot\frac{\hbar c i}{4}[6Q_a j^a + \frac{i}{2}V_d j^{5d}] \\
&= -g\cdot\frac{\hbar c i}{4}[6\cdot\frac{il_0^2}{4}(j_a + k_a)j^a + \frac{i}{2}\cdot 3l_0^2(j^5{}_d + k^5{}_d)j^{5d}] \\
&= +g\cdot\frac{3}{8}\frac{l_0^4}{k}(k_a j^a + k^5{}_d j^{5d})\ ,
\end{aligned} \tag{B.13}$$

where we have used $l_0^2 = \hbar c k$. Similar results hold for the other matter Lagrangian $\mathcal{L}_{m\chi}$. Adding both partial Lagrangians we obtain

$$\begin{aligned}\mathcal{L}_m = {} & g\cdot\hbar c[i\overline{\psi}\gamma^\mu(\nabla^*_\mu+\varepsilon_1 S_\mu)\psi - \frac{mc}{\hbar}\overline{\psi}\psi] + g\cdot\hbar c[i\overline{\chi}\gamma^\mu(\nabla^*_\mu+\varepsilon_2 S_\mu)\chi - \frac{nc}{\hbar}\overline{\chi}\chi] \\ & + g\cdot\frac{3}{8}\frac{l_0^4}{k}(k_a j^a + k^5{}_d j^{5d}) + g\cdot\frac{3}{8}\frac{l_0^4}{k}(j_a k^a + j^5{}_d k^{5d})\ .\end{aligned} \tag{B.14}$$

Next, we insert (B.9) into $\mathcal{L}_G$ and obtain

$$\mathcal{L}_G = -\frac{g}{2k}R = -\frac{g}{2k}[R^* + 3\nabla^*_\mu(Q^\mu - U^\mu) + 3(U_\mu U^\mu + Q_\mu Q^\mu + 4U_\mu Q^\mu) + \frac{1}{24}V_\mu V^\mu]\ . \tag{B.15}$$

This result can be found in [Hor 94]. It can be verified in a cumbersome computation that the derivative term $3\nabla^*_\mu(Q^\mu - U^\mu)$ vanish, if we take into account the Dirac equations for ψ and χ. Then, by inserting (B.10) into (B.15), we get

$$\mathcal{L}_G = -\frac{g}{2k}R^* - g\cdot\frac{3}{8}\frac{l_0^4}{k}(k_\mu j^\mu + k^5{}_\mu j^{5\mu})\ . \tag{B.16}$$

For the third Lagrangian part $\mathcal{L}_Y$, we have simply

$$\mathcal{L}_Y = g\frac{l^2}{4k}S_{\mu\nu}S^{\mu\nu}\ . \tag{B.17}$$

Finally, adding the results (B.14), (B.16) and (B.17) yields exactly (4.24).

B.3 Spinorial algebra

The plane wave spinors used in chapter 4 are taken from [Nac 90] but with a little change in the normalization constant. We first define

$$\chi_{\frac{1}{2}} = \begin{pmatrix}1\\0\end{pmatrix}\ ,\quad \chi_{-\frac{1}{2}} = \begin{pmatrix}0\\1\end{pmatrix}\ ,\quad \varepsilon = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\ . \tag{B.18}$$

Then,

$$u_r(p) = \sqrt{\frac{p^0+m}{2mV}}\begin{pmatrix}\chi_r\\ \frac{\vec{\sigma}\vec{p}}{p^0+m}\chi_r\end{pmatrix}\ ; \tag{B.19a}$$

$$v_r(p) = -\sqrt{\frac{p^0+m}{2mV}}\begin{pmatrix}\frac{\vec{\sigma}\vec{p}}{p^0+m}\varepsilon\chi_r\\ \varepsilon\chi_r\end{pmatrix}\ ; \tag{B.19b}$$

$$s_r(p) = \sqrt{\frac{p^0+n}{2nV}}\begin{pmatrix}\chi_r\\ \frac{\vec{\sigma}\vec{p}}{p^0+n}\chi_r\end{pmatrix}\ ; \tag{B.19c}$$

$$w_r(p) = -\sqrt{\frac{p^0+n}{2nV}}\begin{pmatrix}\frac{\vec{\sigma}\vec{p}}{p^0+n}\varepsilon\chi_r\\ \varepsilon\chi_r\end{pmatrix}\ . \tag{B.19d}$$

The use of the 2×2-matrix ε results in the correct spin directions of the anti-matter waves: All spinors above have exactly the same spin direction determined by p and r, see [Nac 90].

We shall now derive the formula (4.33). Using the Dirac representation (B.1) of the γ-matrices and (B.19a) we have

$$\bar{u}_r(p)\gamma^5\gamma^\mu u_r(p) = \frac{p^0+m}{2mV}(-\chi_r^\dagger\frac{\vec{\sigma}\vec{p}}{p^0+m}, \chi_r^\dagger)\gamma^\mu \begin{pmatrix} \chi_r^\dagger \\ \frac{\vec{\sigma}\vec{p}}{p^0+m}\chi_r^\dagger \end{pmatrix} . \tag{B.20}$$

Note that there are no differences between the anholonomic γ-matrices γ^a and the holonomic ones γ^μ, since we are working in flat Minkowski spacetime. We first derive the 0-th component

$$\begin{aligned} \bar{u}_r(p)\gamma^5\gamma^0 u_r(p) &= \frac{p^0+m}{2mV}(-\chi_r^\dagger\frac{\vec{\sigma}\vec{p}}{p^0+m}, -\chi_r^\dagger)\begin{pmatrix} \chi_r \\ \frac{\vec{\sigma}\vec{p}}{p^0+m}\chi_r \end{pmatrix} \\ &= -\frac{1}{mV}\chi_r^\dagger \begin{pmatrix} p^3 & p^1-ip^2 \\ p^1+ip^2 & -p^3 \end{pmatrix}\chi_r \\ &= \mp\frac{p^3}{mV} \quad \cdots \quad r=\pm\frac{1}{2} . \end{aligned} \tag{B.21}$$

Next, we consider the other three components:

$$\begin{aligned} \bar{u}_r(p)\gamma^5\vec{\gamma}u_r(p) &= \frac{p^0+m}{2mV}(-\chi_r^\dagger\frac{\vec{\sigma}\vec{p}}{p^0+m}, \chi_r^\dagger)\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}\begin{pmatrix} \chi_r \\ \frac{\vec{\sigma}\vec{p}}{p^0+m}\chi_r \end{pmatrix} \\ &= -\chi_r^\dagger\frac{(\vec{\sigma}\vec{p})\vec{\sigma}(\vec{\sigma}\vec{p})}{2mV(p^0+m)}\chi_r - \frac{p^0+m}{2mV}\chi_r^\dagger\vec{\sigma}\chi_r . \end{aligned} \tag{B.22}$$

Using this identity we obtain for the first component

$$\begin{aligned} (\vec{\sigma}\vec{p})\sigma^1(\vec{\sigma}\vec{p}) &= \begin{pmatrix} p^3 & p^1-ip^2 \\ p^1+ip^2 & -p^3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p^3 & p^1-ip^2 \\ p^1+ip^2 & -p^3 \end{pmatrix} \\ &= \begin{pmatrix} 2p^1p^3 & -(p^3)^2+(p^1-ip^2)^2 \\ -(p^3)^2+(p^1-ip^2)^2 & -2p^1p^3 \end{pmatrix} \quad \Rightarrow \\ \bar{u}_r(p)\gamma^5\gamma^1 u_r(p) &= \mp\frac{p^1p^2}{mV(p^0+m)} \quad \cdots \quad r=\pm\frac{1}{2} . \end{aligned} \tag{B.23}$$

Similar considerations for the other two space-directions lead to the result (4.33).

Note that $\overline{\psi}\gamma^5\gamma^\mu\psi$ is always a space-like vector. Indeed, we obtain for the special expression (4.32)

$$\begin{aligned} p^{5\mu}p^5_\mu &= (p^3)^2 - \frac{(p^3)^2\cdot\vec{p}^2}{(p^0+M)^2} - M^2 - \frac{2M(p^3)^2}{(p^0+M)} \\ &= (p^3)^2 - (p^3)^2\frac{(p^0)^2-M^2}{(p^0+M)^2} - M^2 - \frac{2M(p^3)^2}{(p^0+M)} \\ &= -M^2 . \end{aligned} \tag{B.24}$$

For the calculations of the interaction energies, the following identity has been be used (compare (B.19))

$$\bar{u}_r(p)\gamma^5 u_r(p) = \frac{p^0+m}{2mV}(\chi_r^\dagger, \chi_r^\dagger \frac{\vec{\sigma}\vec{p}}{p^0+m}) \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \chi_r \\ \frac{\vec{\sigma}\vec{p}}{p^0+m}\chi_r \end{pmatrix} = 0 \,. \tag{B.25}$$

B.4 Expectation values

We shall now calculate the expectation values of $\widehat{H}_I$ and $\widehat{H}_{ECT}$. First we consider the Einstein–Cartan case with identical particles (4.41d): Introducing the abbreviation

$$\widetilde{dp} := \frac{d^3\!p}{(2\pi)^3}\frac{m}{p^0} \tag{B.26}$$

we obtain

$$\begin{aligned}
\langle 1o|\widehat{H}_{ECT}|1o\rangle &= -\frac{3}{8}\frac{l_0^4}{k}\int_V \langle 0|a_{r'q'}a_{rq} : \overline{\psi}\gamma^5\gamma^\mu\psi\,\overline{\psi}\gamma^5\gamma_\mu\psi : a^\dagger_{rq}a^\dagger_{r'q'}|0\rangle \\
&= -\frac{3}{8}\frac{l_0^4}{k}\int_V \iiiint \widetilde{dp_1}\cdots\widetilde{dp_4} \\
&\quad \langle 0|a_{r'q'}a_{rq} \; : a^\dagger_{r_1p_1}\bar{u}_{r_1}(p_1)e^{ip_1x}\gamma^5\gamma^\mu a_{r_2p_2}u_{r_2}(p_2)e^{-ip_2x}\cdot \\
&\qquad a^\dagger_{r_3p_3}\bar{u}_{r_3}(p_3)e^{ip_3x}\gamma^5\gamma_\mu a_{r_4p_4}u_{r_4}(p_4)e^{-ip_4x} \; : \; a^\dagger_{rq}a^\dagger_{r'q'}|0\rangle \\
&= +\frac{3}{8}\frac{l_0^4}{k}\int_V \iiiint \widetilde{dp_1}\cdots\widetilde{dp_4} \\
&\quad \langle 0|a_{r'q'}a_{rq}a^\dagger_{r_1p_1}a^\dagger_{r_3p_3}a_{r_2p_2}a_{r_4p_4}a^\dagger_{rq}a^\dagger_{r'q'}|0\rangle\cdot \\
&\quad \bar{u}_{r_1}(p_1)\gamma^5\gamma^\mu u_{r_2}(p_2)\bar{u}_{r_3}(p_3)\gamma^5\gamma_\mu u_{r_4}(p_4)\cdot e^{i(p_1-p_2+p_3-p_4)x} \\
&= \frac{3}{8}\frac{l_0^4}{k}V[\;\; 2\bar{u}_r(q)\gamma^5\gamma^\mu u_{r'}(q')\bar{u}_{r'}(q')\gamma^5\gamma_\mu u_r(q) \\
&\qquad - 2\bar{u}_r(q)\gamma^5\gamma^\mu u_r(q)\bar{u}_{r'}(q')\gamma^5\gamma_\mu u_{r'}(q')] \,.
\end{aligned} \tag{B.27}$$

To obtain the last line, we have used the anti-commutation relations of the creation- and annihilation-operators (4.34). We use the following special case of the Fierz transformation (see e.g. [Itz 80]),

$$\overline{\psi}\gamma^5\gamma^\mu\chi\,\overline{\chi}\gamma^5\gamma_\mu\psi = -\overline{\psi}\psi\,\overline{\chi}\chi - \frac{1}{2}\overline{\psi}\gamma^\mu\psi\,\overline{\chi}\gamma_\mu\chi + \overline{\psi}\gamma^5\psi\,\overline{\chi}\gamma^5\chi - \frac{1}{2}\overline{\psi}\gamma^5\gamma^\mu\psi\,\overline{\chi}\gamma^5\gamma_\mu\chi \,, \tag{B.28}$$

and the identity (B.25) to reexpress the last line (B.27) in the following way:

$$\langle 1o|\widehat{H}_{ECT}|1o\rangle = \frac{3}{4}\frac{l_0^4}{k}V[\bar{u}_r(q)\gamma^5\gamma^\mu u_r(q)\bar{u}_{r'}(q')\gamma^5\gamma_\mu u_{r'}(q') - \bar{u}_r(q)u_r(q)\bar{u}_{r'}(q')u_{r'}(q')$$

$$
\begin{aligned}
& \quad - \frac{1}{2}\bar{u}_r(q)\gamma^\mu u_r(q)\bar{u}_{r'}(q')\gamma_\mu u_{r'}(q') \\
& \quad - \frac{1}{2}\bar{u}_r(q)\gamma^5\gamma^\mu u_r(q)\bar{u}_{r'}(q')\gamma^5\gamma_\mu u_{r'}(q')] \\
= & \ -\frac{3}{8}\frac{l_0^4}{k}V[2\bar{u}_r(q)u_r(q)\bar{u}_{r'}(q')u_{r'}(q') + \bar{u}_r(q)\gamma^\mu u_r(q)\bar{u}_{r'}(q')\gamma_\mu u_{r'}(q') \\
& \quad + 3\bar{u}_r(q)\gamma^5\gamma^\mu u_r(q)\bar{u}_{r'}(q')\gamma^5\gamma_\mu u_{r'}(q')] \\
= & \ -\frac{3}{8}\frac{l_0^4}{k}V[2V^{-2} + V^{-2}\frac{q^\mu}{m}\frac{q'_\mu}{m} + 3V^{-2}\frac{q^{5\mu}}{m}\frac{q'^5_\mu}{m}\sigma(r,r')] \ , \qquad \text{(B.29)}
\end{aligned}
$$

where (4.30) to (4.33) were used in the last line. This expression is equal to (4.41d).

The computations of all other expectation values in (4.40) and (4.41) are similar to this example and are therefore left out.

To obtain the expressions (4.42) and (4.43) we need the following relation valid for small momenta $|\vec{q}|, |\vec{q'}| \ll m$:

$$
\begin{aligned}
q^\mu q'_\mu &= \sqrt{m^2 + \vec{q}^2}\sqrt{m^2 + \vec{q'}^2} - \vec{q}\vec{q'} \\
&\approx (m + \frac{\vec{q}^2}{2m})(m + \frac{\vec{q'}^2}{2m}) - \vec{q}\vec{q'} \\
&\approx m^2 + \frac{1}{2}(\vec{q} - \vec{q'})^2 \ . \qquad \text{(B.30)}
\end{aligned}
$$

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